STUDIES OF SENSITIVITY IN LINEAR FEEDBACK CONTROL SYSTEMS

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IN LINEAR FEEDBACK CONTROL SYSTEMS

by

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ABSTRACT

This thesis is the result of literary research on the subject of sensitivity in linear feedback control systems. It is a synopsis of information obtained from various technical publications and is designed to give the reader information on the theory of sensitivity and its application in design problems. Included are

(1) Definition of sensitivity function and root sensitivity for incremental variations of single parameter and their design application.

(2) Definition of sensitivity function applied to the large parameter variations and its design application.

(3) Special topics such as specification for sensitivity function, sensitivity integrals, and introduction to multi-parameter sensitivity.

TABLE OF CONTENTS

TABLE OF ILLUSTRATIONS

TABLE OF SYMBOLS

G	Forward Transfer Function
L	Open Loop Transfer Function
H	Feedback Transfer Function
T	Closed Loop Transfer Function
s	Complex Laplacian Variable
S	Sensitivity function in General
$S_K^{s_j}$	Sensitivity of s_j with Respect to K
σ	Real part of s
w	Imaginary Part of s
K	The System Root Locus Gain
k	General Variable
Σ	Summation Symbol
Π	Product Symbol
P	Pole
Z	Zero

I

INTRODUCTION

One of the foremost properties of feedback is its ability to reduce the sensitivity of a system to variations of the system parameters. Suppose there is an element in the system which must be used because of its special properties. However the element is sensitive to environmental conditions (such as temperature and pressure), or is subjected to aging, or has wide manufacturing tolerances such that when it is replaced the parameters of the new element may be markedly different from those of the old element. Whatever the reason may be, suppose the values of the element parameters vary significantly. It is desired that despite these variations certain system properties, such as the input-output transfer function or output impedance should remain substantially constant. The troublesome element could perhaps be replaced by a better one, but this may be very expensive or impossible. In any case we would like at least to have an alternative to replacing the element. In most cases the troublesome elements are the active elements, vacuum tubes, transistors, energy conversion devices such as motors and generators. But they may also be passive elements or transducers. In general the system assumed in an analytical study will never precisely match the actual physical system; it is therefore important to know the effects of possible variations.

A general knowledge of the effects of parameter changes can be used as a guide to system modifications which would improve the overall performance. For all these reasons measures of closed loop system sensitivity to open loop system parameter variations are an integral part of the analysis problem.

Notions about system sensitivity were in the forefront when the feedback concept was initially developed. This was natural, even unavoidable, since feedback systems possess the fundamental physical property that the effects of variation in the forward loop, whether they are taken as changes in open loop transfer function G(s), or as departures from strict linearity or from freedom from extraneous noise, are reduced by the factor 1 + G in comparison with the effects which would be observed in a non feedback system. Accordingly, sensitivity measures were indispensable to any rational discussion of feedback systems and a useful, classical definition of sensitivity was made. Except for a minor modification the definition of sensitivity, given by Bode, remained unchanged for over a decade. Perhaps this static nature surrounded by dynamic growth in most other areas of feedback systems engineering, made the concept fade with little exposure to sensitivity concepts. In recent years classic sensitivity has become a more popular subject in automatic control and also in network synthesis. Finally, the emphasis on pole-zero specifications for system characteristics gave rise to new concepts of sensitivity, with associated new measures. These measures called here "gain," "open-loop pole," and "open loop zero" sensitivities, were evolved to account for changes in the position of closed loop poles due to shifts or changes in open loop gain, poles or zero.

This thesis is the result of a literary research on the subject of sensitivity in linear feedback control systems. In section II, the system sensitivity and root sensitivity, for a system subjected to incremental parameter variation, are discussed. Based on the meaning of open loop pole and zero sensitivity, a design technique for

compensating the system is also introduced. In Section III, the large parameter variation and a new definition of sensitivity is presented. It is shown that when the leakage transmission between input and output is negligible, the new definition and the classical definition of the system sensitivity are identical. In Section III, an attempt is made to utilize Nichols Charts for evaluation of the sensitivity at a particular frequency range. A design philosophy and technique is also discussed. Section IV discusses some special topics which are of particular interest today and could be selected as interesting topics for further investigation.

II

SENSITIVITY AND INCREMENTAL PARAMETER CHANGES

II-1 - System Sensitivity and Root Sensitivity

Several definitions of sensitivity have been offered in the past. Bode defined the sensitivity of the overall transfer function T to the plant parameter k as

$$S_k^T = \frac{\partial \ln k}{\partial \ln T} = \frac{\frac{\partial k}{k}}{\frac{\partial T}{T}}$$

Horowitz used the inverse of Bode's definition,

$$S_k^T = \frac{\partial \ln T}{\partial \ln k} = \frac{\frac{\partial T}{T}}{\frac{\partial k}{k}}$$

Defined one way or another, S_k^T is generally known as "classical sensitivity" or "system sensitivity," since it involves the system transfer function T. Several other definitions have also been suggested, and each one of these definitions concentrates on some particular engineering or mathematical simplicity. A. Schulke suggests since sensitivity relates changes in the transfer function of a system with respect to changes in parameter of the system, then one can define,

$$\frac{1}{\text{sensitivity}} = \frac{1}{S} = \frac{\partial T}{\partial \ln x}$$

and in general T is a complex and may be expressed as

$$T(x) = A(x) + jB(x)$$

where A(x) is the system attenuation function and B(x) is the system phase function, and from that,

$$\frac{1}{S} = \frac{\partial T}{\partial \ln x} = \frac{\partial}{\frac{\partial x}{x}} \left\{ A(x) + j\, B(x) \right\}$$

$$= x \frac{\partial A(x)}{\partial x} + jx \frac{\partial B(x)}{\partial x}$$

Then assuming that x is real, it is apparent that the real part of $\frac{1}{S}$ is directly related to the system attenuation and that the imaginary part is directly related to the system phase function. The use of S^{-1} which Schulke calls it "sensitiveness" permits comparison of both attenuation and phase functions relative to changes in a common parameter.

The most adopted definition of system sensitivity these days is the one suggested by Horowitz, i. e.

$$S_k^T = \frac{\partial \ln T}{\partial \ln k}$$

and we adopt it through the coming discussions.

In recent years with the increased utilization of the pole-zero approach, it has become increasingly important to examine variations in position of the poles and zeros of the network function due to changes in network parameters. This is known as the "root sensitivity." Formal definitions of root sensitivity vary from author to author. Some define it as,

$$S_x^{s_j} = \frac{\partial s_j}{\frac{\partial x}{x}} = x \frac{\partial s_j}{\partial x}$$

where $(-s_j)$ is the root of the characteristic equation and x a parameter which could be the gain K, or a pole or a zero. Huang uses the definition

$$S_x^{s_j} = \frac{\frac{\partial s_j}{s_j}}{\frac{\partial x}{x}} = \frac{x}{s_j} \frac{\partial s_j}{\partial x}$$

Mc Ruer and Stapleford prefer different definitions for sensitivity to gain, $S_K^{s_j}$, and sensitivity to poles, $S_P^{s_j}$, or sensitivity to zeros, $S_Z^{s_j}$.

Again each one of these definitions leads to a particular engineering or mathematical simplicity utilized by that particular author.

II-2 - Sensitivity Relation in Control Feedback Systems

It has been noted by Bode that for any linear system the transfer function T(s,K), relating a response to an input function can be in the bilinear form of,

$$T(s,K) = \frac{A(s) + KB(s)}{C(s) + KD(s)}$$

and

$$S_K^T = \frac{\partial \ln T}{\partial \ln K} = \frac{\frac{\partial T}{T}}{\frac{\partial K}{K}}$$

Both numerator and denominator of T are of the form,

$$H(s,K) = q(s) + Kp(s)$$

where K is a variable parameter and p(s) and q(s) are polynomials. The roots of $H(s,K) = 0$, which characterize H, depend on the value of K. $H(s,K) = 0$ constitutes an implicit relationship between s and K, and as K takes on all positive values, a number of curves in the S-plane are obtained. These curves are the loci of the roots of H called the "root loci."

The root locus was introduced by Evans in 1948. He used it mainly for the investigation of control systems with a transfer function T, in the form of,

$$T = \frac{G}{1 + G} = \frac{Kp(s)}{q(s) + Kp(s)}$$

where $G = Kp/q$ is called the forward gain in a unity feedback control system. For such systems, only the denominator is of interest. As the result, there arose the generally accepted terminology in which the zeros of p and q were called open loop "poles" and "zeros," respectively and those of $1 + G = 0$ are called "closed loop poles."

II-3 - Relationship Between System Sensitivity and Pole Zero Sensitivities

The transfer function T(s) can generally be written in the form of

$$T(s) = \frac{g\prod_{1}^{n}(s+Z_j)}{\prod_{1}^{m}(s+s_j)} \tag{2-1}$$

and let Z_j, s_j and g be functions of some parameter x (such as temperature, gain of some element, etc.). Then

$$\ln T = \ln g + \sum_{1}^{n}\ln(s+Z_j) - \sum_{1}^{m}\ln(s+s_j)$$

and differentiating above we get

$$\frac{dT}{T} = dx\left[\frac{1}{g}\frac{\partial g}{\partial x} + \sum_{1}^{n}\frac{\partial Z_j}{\partial x}\frac{1}{s+Z_j} - \sum_{1}^{m}\frac{\partial s_j}{\partial x}\frac{1}{s+s_j}\right]$$

dividing both sides by $\frac{dx}{x}$

$$S_x^T = \frac{\frac{dT}{T}}{\frac{dx}{x}} = \frac{\frac{\partial g}{g}}{\frac{\partial x}{x}} + \sum_{1}^{n} x\frac{\partial Z_j}{\partial x}\frac{1}{s+Z_j} - \sum_{1}^{m} x\frac{\partial s_j}{\partial x}\frac{1}{s+s_j} \tag{2-2}$$

Equation (2-2) gives the relationship between the classical definition of sensitivity and the closed loop pole and zero sensitivity, where the closed loop pole and zero sensitivities are defined as

$$S_x^{s_j} = x\frac{\partial s_j}{\partial x} \tag{2-3}$$

$$S_x^{z_j} = x\frac{\partial Z_j}{\partial x} \tag{2-4}$$

This definition is not necessarily the most useful definition at all situations.

Pole Sensitivity (closed loop pole):

The root locus is defined by:

$$H(s,K) \stackrel{D}{=} q(s) + Kp(s) = 0 \tag{2-5}$$

hence·

$$K = -\frac{q(s)}{p(s)} = -\frac{\prod_{j=1}^{n}(s + p_j)}{\prod_{i=1}^{m}(s + Z_i)} \tag{2-6}$$

and

$$\ln[K] = \sum_{j=1}^{n} \ln(s + p_j) - \sum_{i=1}^{m} \ln(s + Z_i) + \pi j \tag{2-7}$$

and

$$\frac{dK}{K} = ds\left[\sum_{j=1}^{n} \frac{1}{(s + p_j)} - \sum_{i=1}^{m} \frac{1}{(s + Z_i)}\right] \tag{2-8}$$

Also since

$$K = -\frac{q(s)}{p(s)}$$

then

$$\frac{dK}{ds} = -\frac{q'p - p'q}{p^2} \tag{2-9}$$

and

$$\frac{\frac{dK}{ds}}{K} = -\frac{\frac{q'p - p'q}{p^2}}{q} = \frac{q'}{q} - \frac{p'}{p} \tag{2-10}$$

For simplicity of notation Equations (2-6) and (2-8) can be written as,

$$K = -\prod_i (s + s_i)^{A_i} \tag{2-11}$$

$$\frac{dK}{K} = ds\left[\sum_i \frac{A_i}{s + s_i}\right] \tag{2-12}$$

which has to be evaluated at $s = -s_j$. The roots of $H = 0$ are denoted as $(-s_j)$ to distinguish them from the variable s which may of course assume any value. In other words $-s_j$ is a point on the root locus.

And

$A_i = 1$ if $-s_i$ is a root of $q(s)$.

$A_i = -1$ if $-s_i$ is a root of $p(s)$.

From Equations (2-10) and (2-12) we can write,

$$S_K^{s_j} = \frac{ds_j}{\frac{dK}{K}} = \left. \frac{1}{\sum_i \frac{A_i}{s + s_i}} \right|_{s=s_j} = \frac{-pq}{qp' - pq'}$$

$$= - \frac{Kp}{q' + Kp'} \qquad (2\text{-}13)$$

where $S_K^{s_j}$ is the closed loop pole sensitivity with respect to K and sometimes referred to as just pole sensitivity with respect to K.

Furthermore the residue of a function $\frac{A(s)}{B(s)}$ at pole $s = a$ is defined as

$$\underset{\text{at } s=a}{\text{Residue}} \; \frac{A(s)}{B(s)} = \lim_{s \to a} \frac{(s - a)\, A(s)}{B(s)}$$

$$= \lim_{s \to a} \frac{(s - a)\, A'(s) + A(s)}{B'(s)}$$

$$= \lim_{s \to a} \frac{A(s)}{B'(s)}$$

which is true for simple poles. Noting that $\frac{\partial H}{\partial s_j} = q' + Kp'$ and applying above residue theorem we get

$$S_K^{s_j} = - \frac{Kp}{q' + Kp'} = - \frac{Kp}{\frac{\partial H}{\partial s_j}} = -Q_j \qquad (2\text{-}14)$$

where Q is the residue of

$$T(s) = \frac{Kp(s)}{q(s) + Kp(s)}$$

at point $s = -s_j$. The same result can be obtained directly from (2-5)

$$S_K^{s_j} = \frac{ds_j}{\frac{dK}{K}} = - K \frac{\frac{\partial H}{\partial K}}{\frac{\partial H}{\partial s_j}} \qquad (2\text{-}15)$$

and since $H = q + Kp$, then

$$S_K^{s_j} = -K\frac{\frac{\partial H}{\partial K}}{\frac{\partial H}{\partial s_j}} = -\left.\frac{Kp}{\frac{dq}{ds} + K\frac{dp}{ds}}\right|_{s=-s_j} = -\left.\frac{Kp}{\frac{\partial H}{\partial s}}\right|_{s=-s_j} = -Q_j$$

The relation $S_K^{s_j} = -Kp/(q' + Kp')$ is quite convenient since it only requires knowledge of the "open loop" transfer function and need not be in factored form. The evaluation of the pole sensitivity $S_K^{s_j}$ by the residue of the closed loop transfer function will call for the complete determination of the closed loop transfer function.

II-4 - Slope of Root Locus

Slope of the root locus is the tangent of the argument of s_j and since,

$$S_K^{s_j} = \frac{\Delta s_j}{\frac{\Delta K}{K}} \tag{2-16}$$

Then $$\Delta s_j = S_K^{s_j}\frac{\Delta K}{K} \tag{2-17}$$

Since $\frac{\Delta K}{K}$ is a positive number, then

$$\angle \Delta s_j = \angle S_K^{s_j} \tag{2-18}$$

where $\angle \Delta s_j$ represents arg Δs_j. So

$$\frac{dy}{dx} = \tan\angle \Delta s_j = \tan\angle S_K^{s_j} = \frac{I_m S_K^{s_j}}{\mathrm{Re} S_K^{s_j}} \tag{2-19}$$

But from Equation (2-8) we have,

$$\frac{I_m S_K^{s_j}}{\mathrm{Re} S_K^{s_j}} = \frac{I_m\left[\frac{1}{\sum_{j=1}^{n}\frac{1}{s+p_j} - \sum_{i=1}^{m}\frac{1}{s+z_i}}\right]_{s=-s_j}}{\mathrm{Re}\left[\frac{1}{\sum_{j=1}^{n}\frac{1}{s+p_j} - \sum_{i=1}^{m}\frac{1}{s+z_i}}\right]_{s=-s_j}} \tag{2-20}$$

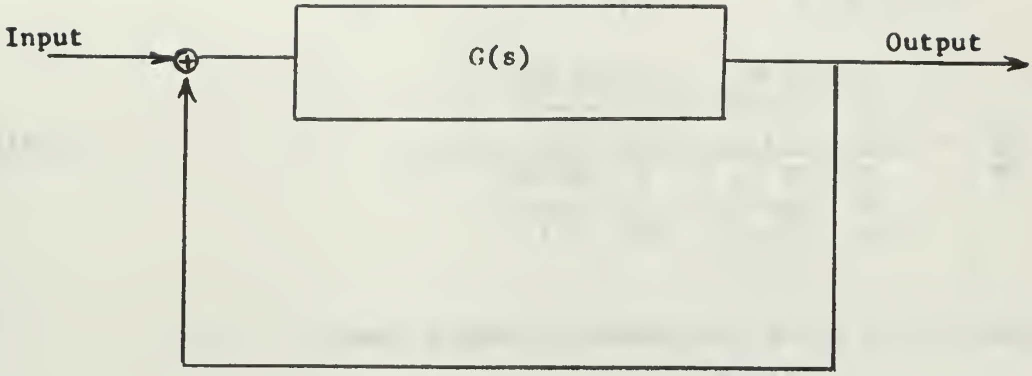


Fig. 2-1: Unity Feedback System.

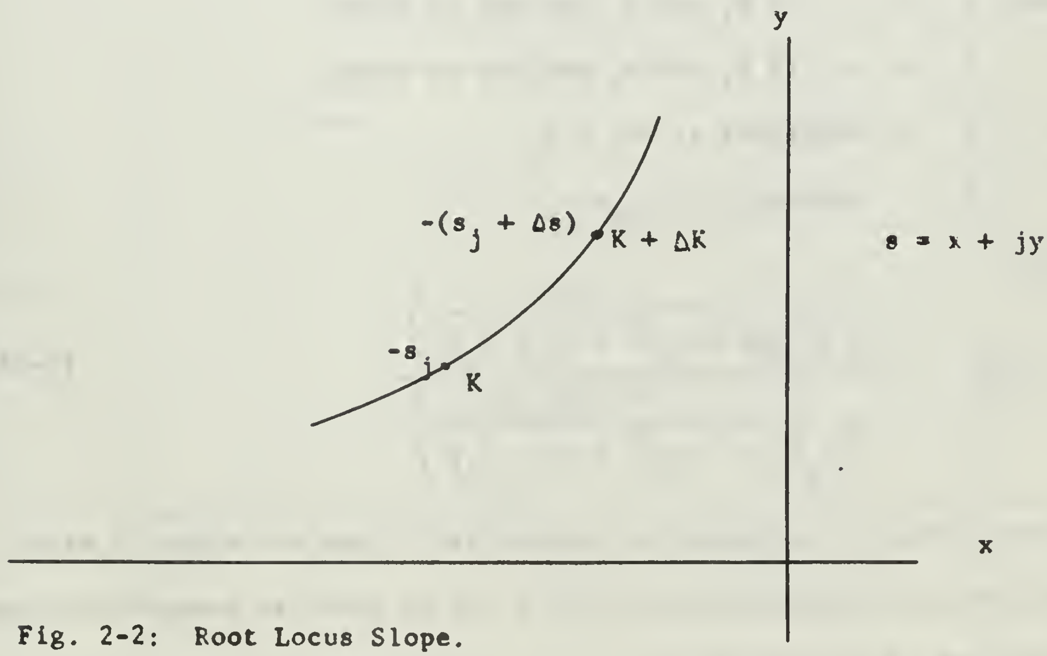


Fig. 2-2: Root Locus Slope.

If we write

$$s + p_j = \ell_j \; e^{j\theta_j}$$

$$s + z_i = \ell_i \; e^{j\theta_i}$$

and since

$$I_m\left(\frac{1}{\sum \frac{1}{s + p_j}}\right) = I_m \frac{1}{\sum \frac{1}{\ell_j} e^{-j\theta_j}} = \sum \frac{\sin \theta_j}{\ell_j}$$

then

$$\frac{dy}{dx} = \frac{\sum_{j=1}^{m} \frac{\sin \theta_j}{\ell_j} - \sum_{i=1}^{n} \frac{\sin \theta_i}{\ell_i}}{\sum_{j=1}^{m} \frac{\cos \theta_j}{\ell_j} - \sum_{i=1}^{n} \frac{\cos \theta_i}{\ell_i}} \tag{2-21}$$

Equation (2-21) could be written in simple form,

$$\frac{dy}{dx} = \frac{\sum A_i \frac{\sin \theta_i}{\ell_i}}{\sum A_i \frac{\cos \theta_i}{\ell_i}} \tag{2-22}$$

where $A_i = 1$ if ℓ_i and θ_i are due to poles

$A_i = -1$ if ℓ_i and θ_i are due to zeros,

and ℓ_i = magnitude of $-s_j + s_i$

θ_i = argument of $-s_j + s_i$

Then

$$\frac{dy}{dx} = \frac{\sum A_i \left(\frac{y - y_i}{(x - x_i)^2 + (y - y_i)^2}\right)}{\sum A_i \left(\frac{x - x_i}{(x - x_i)^2 + (y - y_i)^2}\right)} \tag{2-23}$$

In some cases it is easier to compute dy/dx from the slope of ds/dK obtained by differentiation of $-K = p/q$ as shown in example discussed at the end of this chapter.

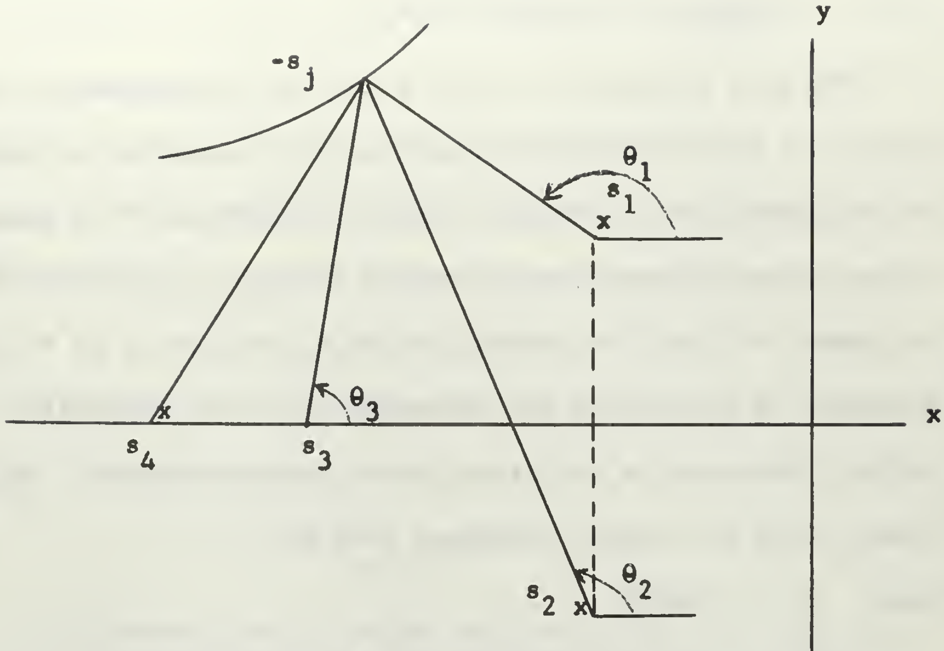


Fig. 2-3: Argument and magnitude of $-s_j - s_i$

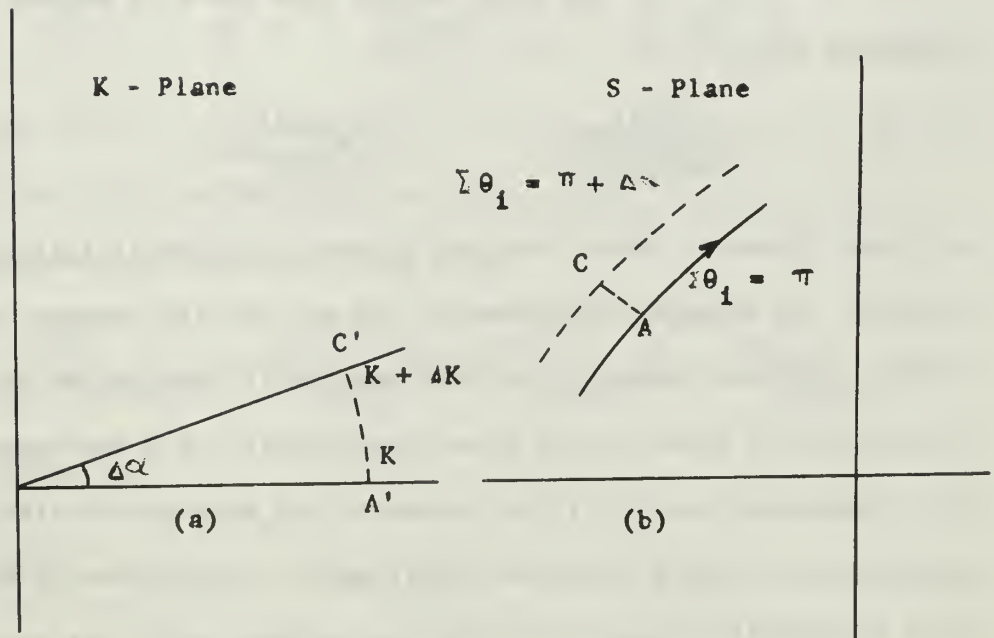


Fig. 2-4: (a) Definition of K

(b) Evaluation of $S_K^{s_j}$

II-5 - Graphical Evaluation of $S_K^{s_j}$

The pole sensitivity at any point "A" corresponding to a particular value of s can be evaluated graphically by summation of angles and without evaluation of K. Choose a point C such that AC is perpendicular to the locus and evaluate the phasor AC = ΔS. Evaluate the value of the phase of K at C by summing angles θ_i and let it be $\pi + \Delta\alpha$. This procedure is based upon the independence of the derivative of an analytic function on the direction of differentiation. Hence ΔK is chosen to be A'C' which is mapped into AC.

Then: $\quad K = j(\Delta\alpha)K$

$$\frac{\Delta s}{j\Delta\alpha} = \frac{\Delta s}{j\,\frac{\Delta K}{jK}} = \frac{\Delta s}{\frac{\Delta K}{K}} = S_K^{s_j} \qquad (2\text{-}24)$$

Therefore knowing the pahsor Δs and the angle $\Delta\alpha$ one can evaluate $S_K^{s_j}$.

II-6 - Curvature, Pole Sensitvity and Their Relation

The curvature at any point on the root locus is proved to be (Appendix II),

$$\frac{1}{\rho} = -\frac{\mathrm{Im}(\ln K)_{zz}}{\mathrm{Im}\,j(\ln K)_z} = -\frac{\mathrm{Im}(\ln K)_{zz}}{\mathrm{Re}(\ln K)_z} \qquad (2\text{-}25)$$

At first glance it seems that the curvature and sensitivity are somehow related. An example, discussed at the end of this chapter, will reveal, however that the curvature in that example is constant while the pole sensitivity is equal to cot ϕ and consequently is a continuous variable. The independent nature of the curvature and pole sensitivity can be understood by noting Equation (2-25) and the definition of sensitivity. Pole sensitivity depends on first derivatives, while the curvature depends on the second derivatives, and in general the two derivatives are independent. As suggested by Ur we can think of curvature as

"kinematic" property and the pole sensitivity as a "dynamic property."

II-7 - Open Loop Pole and Zero Sensitivity (Figure 2-5)

Let $G(s)$ be the forward transfer function and L be the loop transfer function. Then $L = GF$ and the system characteristic equation is

$$1 + L = 0$$

and if $-s_j$ is the system root, then

$$1 + L(s)\Big|_{s=-s_j} = 0$$

where $s = -s_j$ represents the point on the root locus. If K is the gain constant of $L(s)$ and Z_i, P_i its open loop seros and poles, then we can write:

$$L = L(s, K, Z_i, P_i)$$

and taking the total differential of L:

$$dL = \frac{\partial L}{\partial s} dS + \frac{\partial L}{\partial K} dK + \sum_{i=1}^{n} \frac{\partial L}{\partial Z_i} dZ_i + \sum_{i=1}^{m} \frac{\partial L}{\partial P_i} dP_i \tag{2-26}$$

On the root locus $1 + L = 0$ or $L(s) = -1$ and therefore $L(s) =$ constant, and the total differential dL is zero for $s = -s_j$. Then letting $dL = 0$ and $s = -s_j$, Equation (2-26) gives,

$$0 = \frac{\partial L}{\partial s}\Big|_{s=-s_j} ds_j + \frac{\partial L}{\partial K}\Big|_{s=-s_j} dK + \sum \frac{\partial L}{\partial Z_i}\Big|_{s=-s_j} dZ_i + \sum \frac{\partial L}{\partial P_i}\Big|_{s=-s_j} dP_i$$

Rearranging we get

$$ds_j = \frac{1}{\left(\frac{\partial L}{\partial s}\right)_{s=-s_j}} \left[\left(\frac{\partial L}{\partial K}\right)_{s=-s_j} dK + \sum \left(\frac{\partial L}{\partial Z_i}\right)_{s=-s_j} dZ_i + \sum \left(\frac{\partial L}{\partial P_i}\right)_{s=-s_j} dP_i \right] \tag{2-27}$$

But s_j itself is a function of K, Z_i and P_i, so

$$s_j = s_j(K, Z_i, P_i) \tag{2-28}$$

Taking the total differential of s_j we get

$$ds_j = K\frac{\partial s_j}{\partial K}\frac{dK}{K} + \sum_i \frac{\partial s_j}{\partial Z_i} dZ_i + \sum_i \frac{\partial s_j}{\partial P_i} dP_i \tag{2-29}$$

Equation (2-29) suggests that ds_j be written as

$$ds_j = S_K^{s_j}\frac{dK}{K} + \sum S_{Z_i}^{s_j} dZ_i + \sum S_{P_i}^{s_j} dP_i \tag{2-30}$$

Equations (2-29) and (2-30) thus define the sensitivity of root s_j to gain K, as

$$S_K^{s_j} \stackrel{D}{=} \frac{\partial s_j}{\frac{\partial K}{K}} \tag{2-31}$$

which is the same as Equation (2-13) and the sensitivity of root s_j to open loop zero Z_i as

$$S_{Z_i}^{s_j} \stackrel{D}{=} \frac{\partial s_j}{\partial Z_i} \tag{2-32}$$

and the sensitivity of root s_j to open loop pole P_i as

$$S_K^{s_j} \stackrel{D}{=} \frac{\partial s_j}{\partial P_i} \tag{2-33}$$

II-8 - Relationship between $S_K^{s_j}$, $S_Z^{s_j}$, and $S_P^{s_j}$

Comparing Equations (2-27) and (2-30) yields:

$$S_K^{s_j} = K\left(\frac{\partial L/\partial K}{\partial L/\partial S}\right)_{s = -s_j} \tag{2-34}$$

$$S_{Z_i}^{s_j} = \left(\frac{\partial L/\partial Z_i}{\partial L/\partial S}\right)_{s = -s_j} \tag{2-35}$$

$$S_{P_i}^{s_j} = \left(\frac{\partial L/\partial P_i}{\partial L/\partial S}\right)_{s = -s_j} \tag{2-36}$$

But

$$L = K\frac{\prod_1^n (s + Z_i)}{\prod_1^m (s + P_i)} \tag{2-37}$$

Then

$$\left(\frac{\partial L}{\partial K}\right)_{s=-s_j} = \left.\frac{\prod_1^n (s + Z_i)}{\prod_1^m (s + P_i)}\right|_{s = -s_j} = \left(\frac{L}{K}\right)_{s = -s_j} \tag{2-38}$$

But since $1 + L(s)\Big|_{s = -s_j} = 0,$

then $L(s)\Big|_{s = -s_j} = 1$, and Equation (2-38) yields to

$$\left(\frac{\partial L}{\partial K}\right)_{s = -s_j} = -\frac{1}{K} \tag{2-39}$$

Then (2-34) becomes

$$S_K^{s_j} = -1\Big/\left(\frac{\partial L}{\partial s}\right)_{s = -s_j} \tag{2-40}$$

Similarly

$$\left(\frac{\partial L}{\partial Z_i}\right)_{s = -s_j} = \left.\frac{K\prod_1^{n-1} (s + Z_i)}{\prod_1^m (s + P_i)}\right|_{s = -s_j}$$

$$= \left.\frac{L}{s + Z_i}\right|_{s = -s_j} = \frac{-1}{-s_j + Z_i} \tag{2-41}$$

Then,

$$S_{Z_i}^{s_j} = \left(\frac{\partial L/\partial Z_i}{\partial L/\partial S}\right)_{s = -s_j} = \frac{-1/(-s_j + Z_i)}{(\partial L/\partial S)_{s = -s_j}}$$

OR

$$S_{Z_i}^{s_j} = \frac{S_K^{s_j}}{-s_j + Z_i} \tag{2-42}$$

Similarly

$$S_{P_i}^{s_j} = \frac{S_K^{s_j}}{s_j - P_i} \tag{2-43}$$

Equations (2-42) and (2-43) dictate that the sensitivities to all singularities are directly proportional to $S_K^{s_j}$ and inversely proportional to the distance between $-s_j$ and the singularity involved

(in this case $s = -Z_i$ and $S = -P_i$ are the signularities) Thus whatever properties are found for $S_K^{s_j}$ may be extended to $S_Z^{s_j}$ or $S_P^{s_j}$.

A particular case of Equation (2-43) is for $P_o = 0$, i.e. the root sensitivity to the pole at origin of the S-plan.

$$S_{P_o}^{s_j} = \frac{S_K^{s_j}}{(s_j)} \tag{2-44}$$

Then $S_{P_o}^{s_j}$ is proportional to $S_K^{s_j}$, the constant of proportionality being $\frac{1}{s_j}$, a complex quantity.

II-9 - Relationship between Root Sensitivity of S_j and Residue

In Equation (2-14), we proved that in case of unity feedback,

$$S_K^{s_j} = -Q_j \tag{2-45}$$

where Q_j is the residue of $T(s) = \frac{Kp(s)}{q(s) + Kp(s)}$ at point $-s_j$.

In this section it will be shown that if $-s_j$ is a single root, then

$$S_K^{s_j} = F(-s_j)\, Q_j \tag{2-46}$$

where $F(-s_j)$ is the feedback transfer function F evaluated at $s = -s_j$ and Q_j is the residue of the system transfer function at $(-s_j)$. For unity feedback and simple root $F(-s_j) = -1$ and Equation (2-46) yields to Equation (2-45). The overall transfer function T(s) of the system shown in Figure (2-5) is

$$T(s) = \frac{G(s)}{1 + F(s)G(s)} = \frac{F(s)G(s)}{F(s)[1 + F(s)G(s)]} \tag{2-47}$$

But $L(s) = F(s)G(s)$, then

$$T(s) = \frac{L(s)}{F(s)[1 + L(s)]} \tag{2-48}$$

The residue of T(s) at $-s_j$ is

$$Q_j = (s + s_j)T(s)\Big|_{s = -s_j} = \frac{(s + s_j)L(s)}{F(s)[1 + L(s)]}\Bigg|_{s = -s_j}$$

Then, let the rightmost expression be denoted as R_j, i.e. by definition,

$$R_j(-s_j) = Q_j \tag{2-49}$$

Then $$R_j F(1 + L) = (s + s_j)L$$

Taking the derivative with respect to s of both sides,

$$\frac{\partial F}{\partial s}[1 + L]R_j + FR_j\frac{\partial L}{\partial s} + F[1 + L]\frac{\partial R_j}{\partial s} = L + (s + s_j)\frac{\partial L}{\partial s}$$

At $s = -s_j$ i.e. at a point on the root locus, $L = -1$ the above equation reduces to

$$FR_j \frac{\partial L}{\partial s}\bigg|_{s = -s_j} = -1$$

OR $$R_j(-s_j) = -1/F(-s_j)\left[\frac{\partial L}{\partial s}\right]_{s = -s_j} \tag{2-50}$$

Comparing Equations (2-50) and (2-40) we get

$$S_K^{s_j} = F(-s_j)\, Q_j \tag{2-51}$$

Now turning to the case of N order root at $-s_j$ we can expand

$$T(s) = \frac{L}{F(1 + L)}$$

into partial fraction as

$$T(s) = \frac{Q_{j1}}{s + s_j} + \frac{Q_{j2}}{(s + s_j)^2} + \cdots + \frac{Q_{jN}}{(s + s_j)^N} + \text{terms from other roots} \tag{2-52}$$

where

$$Q_{jK} = \frac{1}{(N - K)!}\left[\frac{\partial^{N-K}}{\partial s^{N-K}}\left[\frac{(s + s_j)^K L}{F(1 + L)}\right]\right]_{s = -s_j} \tag{2-53}$$

Again defining the quantity inside the small bracket as R_j, then repeating the operations as before, one obtains after repeated differentiation

$$Q_{jN} = \frac{-N!}{F(-s_j)\left.\dfrac{N_L}{s^N}\right|_{s=-s_j}} \tag{2-54}$$

and from Appendix III

$$S_K^{s_j} = \frac{(-1)^N N!}{\left[\dfrac{\partial N_L}{\partial s^N}\right]_{s=-s_j}} \tag{2-55}$$

Theorem

When $-s_j$ is a single order system root, the sum of the sensitivities of s_j to open loop poles and zeros is equal to unity, i.e.

$$\sum_i S_{Z_i}^{s_j} + \sum_i S_{P_i}^{s_j} = 1 \tag{2-56}$$

This is easily seen by referring to the construction of root loci. If all open loop zeros and poles are displaced by the same amount δ, then all closed loop roots are displaced by the same amount δ, that is if $dZ_i = dP_i = \delta$ for all i, then $ds_j = \delta$ for all i. Therefore if in Equation (2-29) we put $ds_j = dZ_i = dP_i = \delta$ and $dK = 0$ we get

$$\sum \frac{\partial s_j}{\partial Z_i} + \sum \frac{\partial s_j}{\partial P_i} = 1$$

A more rigorous proof is as follows. Rewriting equations (2-13) and (2-42) and (2-43) we get

$$S_{Z_i}^{s_j} = \frac{S_K^{s_j}}{Z_i - s_j}$$

$$S_{P_i}^{s_j} = \frac{S_K^{s_j}}{s_j - P_i}$$

$$S_K^{s_j} = \frac{1}{\left[\sum_{i=1}^{n} \frac{1}{Z_i - s_j} + \sum_{i=1}^{m} \frac{1}{s_j - P_i}\right]}$$

Then

$$\sum_{i=1}^{n} S_{P_i}^{s_j} + \sum_{i=1}^{m} S_{Z_i}^{s_j} = \sum_{i=1}^{n} \frac{S_K^{s_j}}{Z_i - s_j} + \sum_{i=1}^{m} \frac{S_K^{s_j}}{s_j - P_i}$$

$$= S_K^{s_j} \left[\sum_{i=1}^{n} \frac{1}{Z_i - s_j} + \sum_{i=1}^{m} \frac{1}{s_j - P_i}\right]$$

$$= 1$$

The above relation is only valid for simple poles.

II-10 - Graphical Method for Determining Open Loop Pole and Zero Sensitivities

Equations (2-42) and (2-43) suggest that to singularities which are close to system root ($-s_j$), s_j is more sensitive and for singularities which are far away, s_j is less sensitive until it becomes insensitive to singularities at infinity. This concerns the magnitude of sensitivities. But sensitivities are vector quantities, since the change of a parameter may move the roots in different directions. It is then helpful to make use of Equation (2-56) together with Equations (2-42) and (2-43).

Referring to Equations (2-42) and (2-43) where Z and P indicate open loop singularities and $-s_j$ is the system root in question, we can draw a vector from $-s_j$ toward each and every pole, and away from each and every zero. The length of each vector is inversely proportional to the distance from $-s_j$ to the singularity concerned. Then construct the sum U of all these vectors, which is a vector itself, (Figure 2-5). If U is taken as a unity vector in magnitude and phase, then the other vectors measure the sensitivity of s_j to each

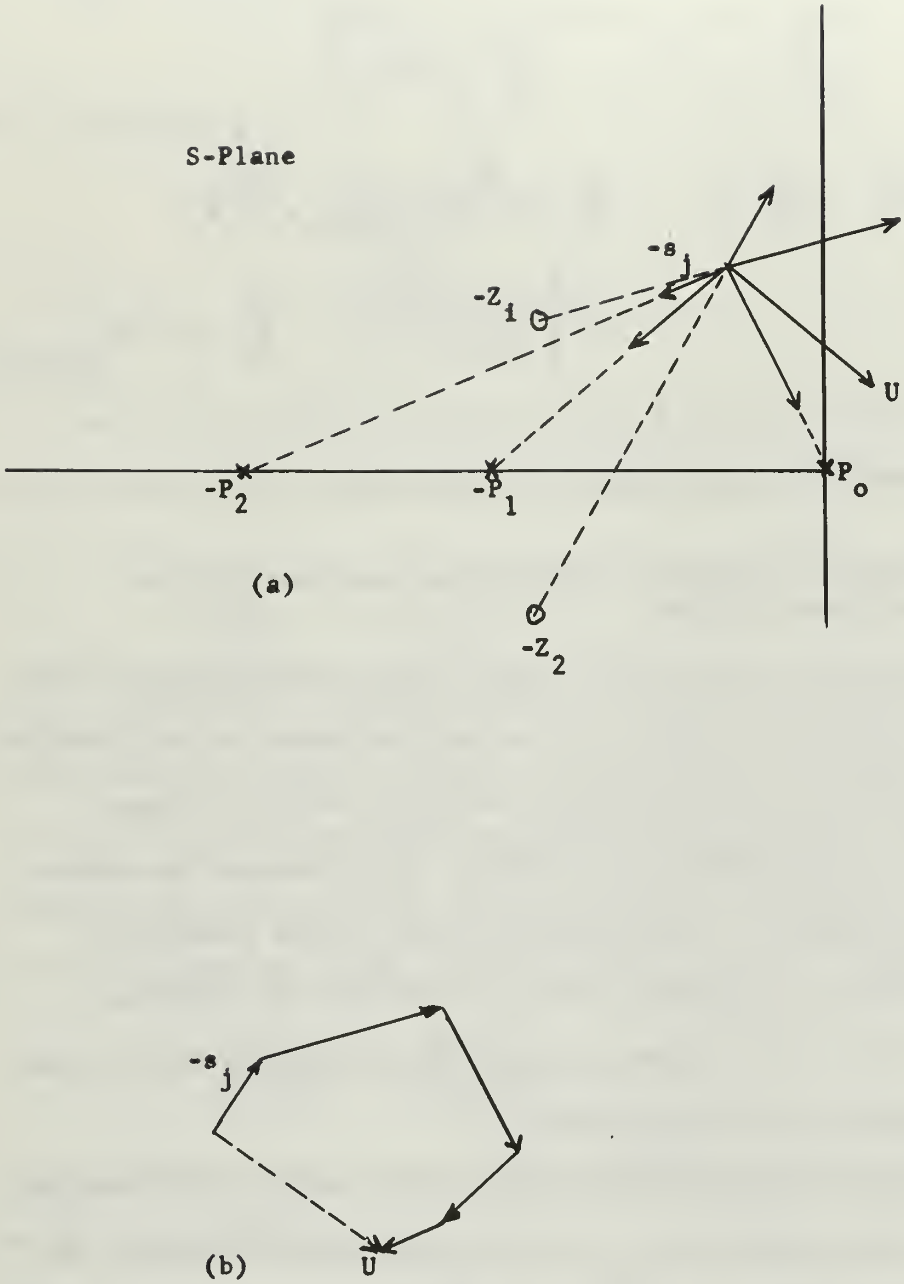


Fig. 2-5: (a) Construction of vector diagram

(b) Vector addition to obtain U.

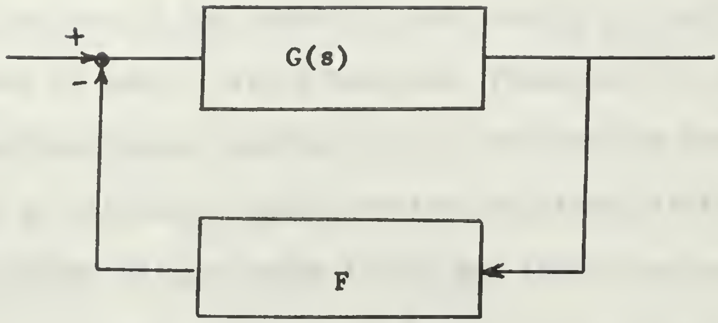


Fig. 2-6: A Basic Feedback System.

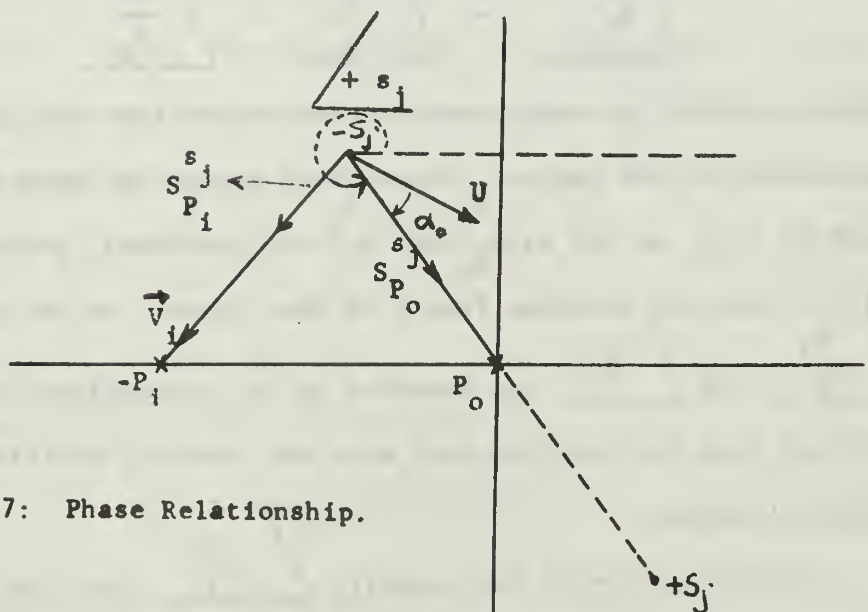


Fig. 2-7: Phase Relationship.

singularity respectively. So we regard it as

$$U = 1\underline{/o} = 1\, e^{jo} \tag{2-57}$$

and is named by Thaler and Rung as "unit-sensitivity" vector. This unity scale does not apply to sensitivity to gain, $S_K^{s_j}$. Actually this technique is quite similar to the graphical method of evaluating the residues for simple singularities, and by putting $U = 1\underline{/o}$ we establish the necessary reference scale. Phase of sensitivity vector is measured as positive in the clockwise sense starting from the U vector. This seemingly arbitrary sign convention in fact comes from Equations (2-42) and (2-43) which are the basis of this vector diagram.

$$S_{P_i}^{s_j} = \frac{S_K^{s_j}}{s_j - P_i} = \frac{S_K^{s_j}}{(-P_i) - (-s_j)} = \frac{S_K^{s_j}}{\vec{V}_i} \tag{2-58}$$

The denominator is the vector from root $(-s_j)$ toward pole $(-P_i)$ as shown in Figure 2-7. The phase relationship of the above equation is

$$\underline{/S_{P_i}^{s_j}} = \underline{/S_K^{s_j}} - \underline{/\vec{V}_i} \tag{2-59}$$

Relation (2-59) is true no matter what conventions are applied to the measurement of the angles. Since the U vector has been found to be equal to $1\underline{/o}$ as far as $S_{P_i}^{s_j}$ and $S_{Z_i}^{s_j}$ are concerned, phase of $S_{P_i}^{s_j}$ will be measured starting from U as zero phase. On the other hand $\underline{/S_K^{s_j}}$ and $\underline{/\vec{V}_i}$ are measured in the conventional way, i.e. starting from the positive real axis and counting positively counterclockwise.

In Equation (2-59) the quantity $\underline{/S_K^{s_j}}$ does not depend on i, i.e. it is the same for all i's. Thus for each i,

$$\underline{/S_{P_i}^{s_j}} = \text{constant} - \underline{/V_i}$$

meaning that the larger the value of $\angle V_i$ the smaller must be $\angle S_{P_i}^{s_j}$. Since $\angle V_i$ is measured conventionally (positive counterclockwise) then $\angle S_{P_i}^{s_j}$ must be measured positively clockwise.

Next we prove that vector $S_K^{s_j}$ lies on U. From Equation (2-44)

$$S_{P_o}^{s_j} = \frac{S_K^{s_j}}{(s_j)}$$

and
$$\angle S_K^{s_j} = \angle S_{P_o}^{s_j} + \angle s_j$$

Figure 2-7 shows the angles $\angle s_j$ and $\angle S_{P_o}^{s_j} = \alpha$, the latter being measured from U. Since as shown in Figure 2-7, $\angle S_{P_o}^{s_j}$ + $\angle s_j$ gives the direction of U on the S-plane, it is thus established that on the S-plane, $S_K^{s_j}$ always lies on U. Since $S_K^{s_j}$ indicates the direction in which the root moves when K varies, i.e. the direction of the root locus, the above result can be stated as: "At any point on the root locus, the U vector is tangent to the root locus." Also Equation (2-59) reduces to

$$\angle S_{P_i}^{s_j} = - \angle V_i \qquad (2\text{-}60)$$

II-11 - Locus of U on the S-Plane

Consider the plant to be compensated,

$$G = \frac{K(s + Z_1)}{s(s + P_1)(s + P_2)} \qquad (2\text{-}61)$$

and the desired dominant-root location $-s_j$ as indicated in Figure 2-8. A spirule measurement shows that an additional phase of $+\phi$ is needed at location $-s_j$. Thus a lead network with a zero at Z and a pole at P is needed. The question is, how does the tip of the U

vector move on the S-plane, when Z and P take all possible values on the negative real axis?

Now sensitivity vectors should be drawn for all the plants poles and zeros, ignoring for the moment, the need for compensation. All these vectors can then be added to obtain the vector sum QI, which may be called the "uncompensated" unity vector. Then the geometric locus of U is a circle with a center I and a radius of $r = \frac{1}{d \sin \phi}$. This circle is named the (U) circle.

<u>Proof</u>:

QI would be the unity vector without "P" and "Z" (Figure 2-8a). When P and Z are added so that $\widehat{PQZ} = \phi$, a sensitivity vector must be assigned to each of these singularities, $Qm = \frac{1}{QP}$ is assigned to the pole and $Qu = \frac{1}{QZ}$ is assigned to the zero. The vector QI then increases by the vector quantities IM = Qm and MU = Qu. QU is the final U vector. Note that, in Figure 2-8a, IN and MU are two equal but opposite vectors.

Figure 2-8b shows that when P moves along the real axis, since $Qm = \frac{1}{QP}$, m moves on a radius of $r = \frac{1}{2d}$, because of inversion of real axis with Q as center and ratio of inversion one. Such a circle is referred to as the (m) circle. If we translate this (m) circle by QI, then we get (M) circle which is the locus of point M. So the locus of point M is the (M) circle, or radius $R = \frac{1}{2d}$ and with I as its upper most point. Figure 2-8c shows that:

$$IU = MN = 2R \sin \frac{\widehat{MON}}{2}$$

$$IU = 2R \sin \phi = \frac{1}{d} \sin \phi = \text{constant} \qquad (2\text{-}62)$$

Hence U moves on a circle centered at I and of radius $r = \frac{1}{d} \sin \phi$. So the following result can be stated:

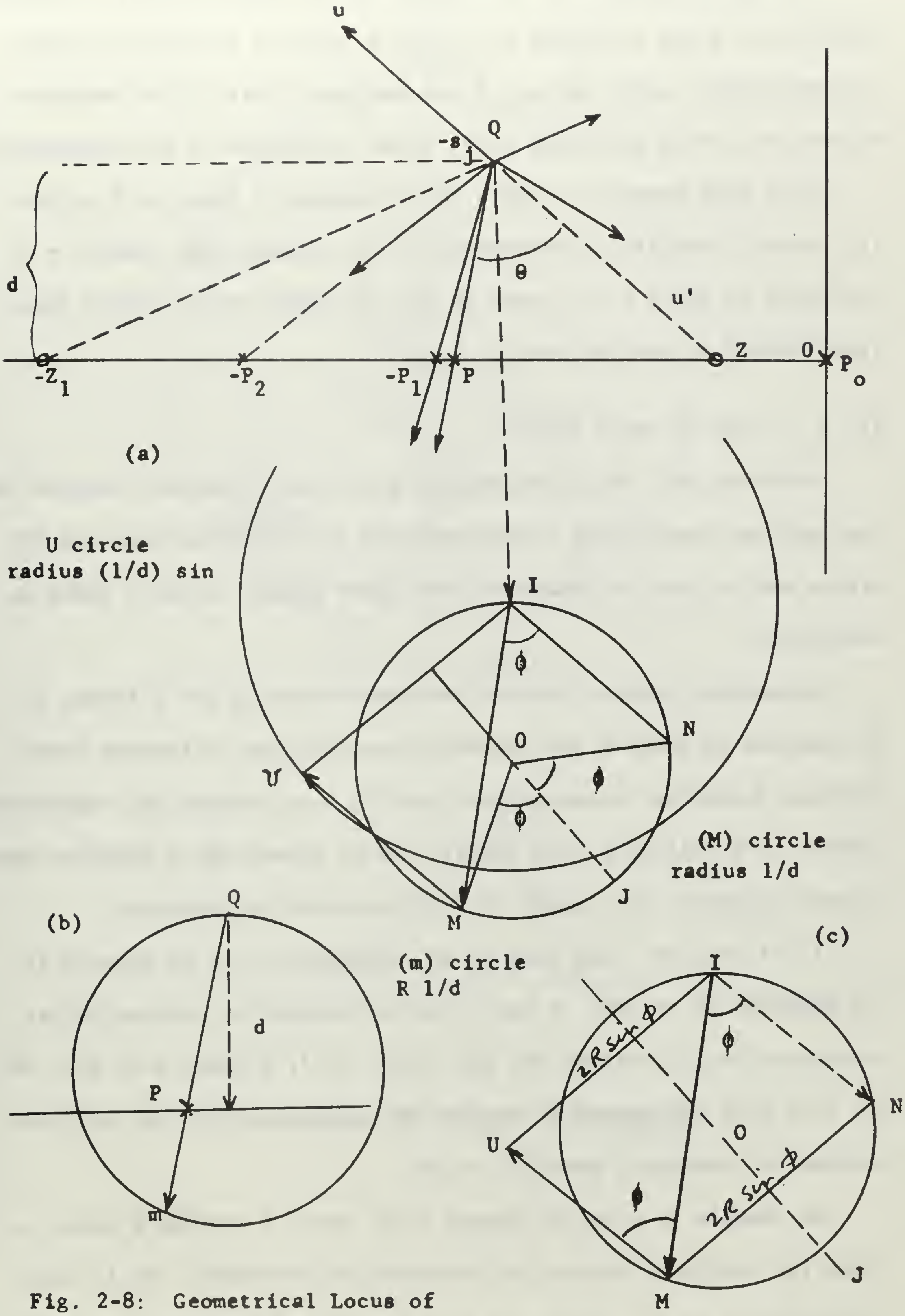


Fig. 2-8: Geometrical Locus of Point U.

"The geometric locus of the tip of the unity vector is a circle, centered at I and of radius $r = \frac{1}{d} \sin \phi$, where I is the tip of the 'uncompensated' unity vector, d the imaginary part of the dominant system root and ϕ the phase shift to be introduced by the compensation."

Note that there are limits to the geometric locus of U on the (U) circle. One limit corresponds to the extreme case where $P = \infty$, the other is for $Z = 0$. Later it will be shown how to obtain these limit points on the (U) circle.

II-12 - "Use of the U Locus"

Starting with the uncompensated plant and a desired location for the dominant root, it is always possible to be able to draw the (U) circle and be able to obtain the two limit points on the U locus on the circle.

Conversely after a point U has been chosen on the U locus, it is possible to perform the graphical construction in reverse order and thus drive the corresponding Z and P, i.e., obtain the compensation needed. The following will explain how to choose the U point on the U locus, which is the seence of this construction technique.

In Figure 28a, note that OJ is perpendicular to IU where J is the midpoint of arc MN. P and Z can be obtained by drawing OJ perpendicular to IU, cutting the (M) circle at J, drawing arcs JM = JN = ϕ. P is then determined by drawing OP parallel to IM, and Z is determined by drawing QZ parallel to IN.

An example is given in Figure 2-9a, where a desired U point is shown (an arbitrary choice for illustrative purposes). OJ is drawn perpendicular to IU. Next, arc JM = arc JN = ϕ is drawn. In Figure

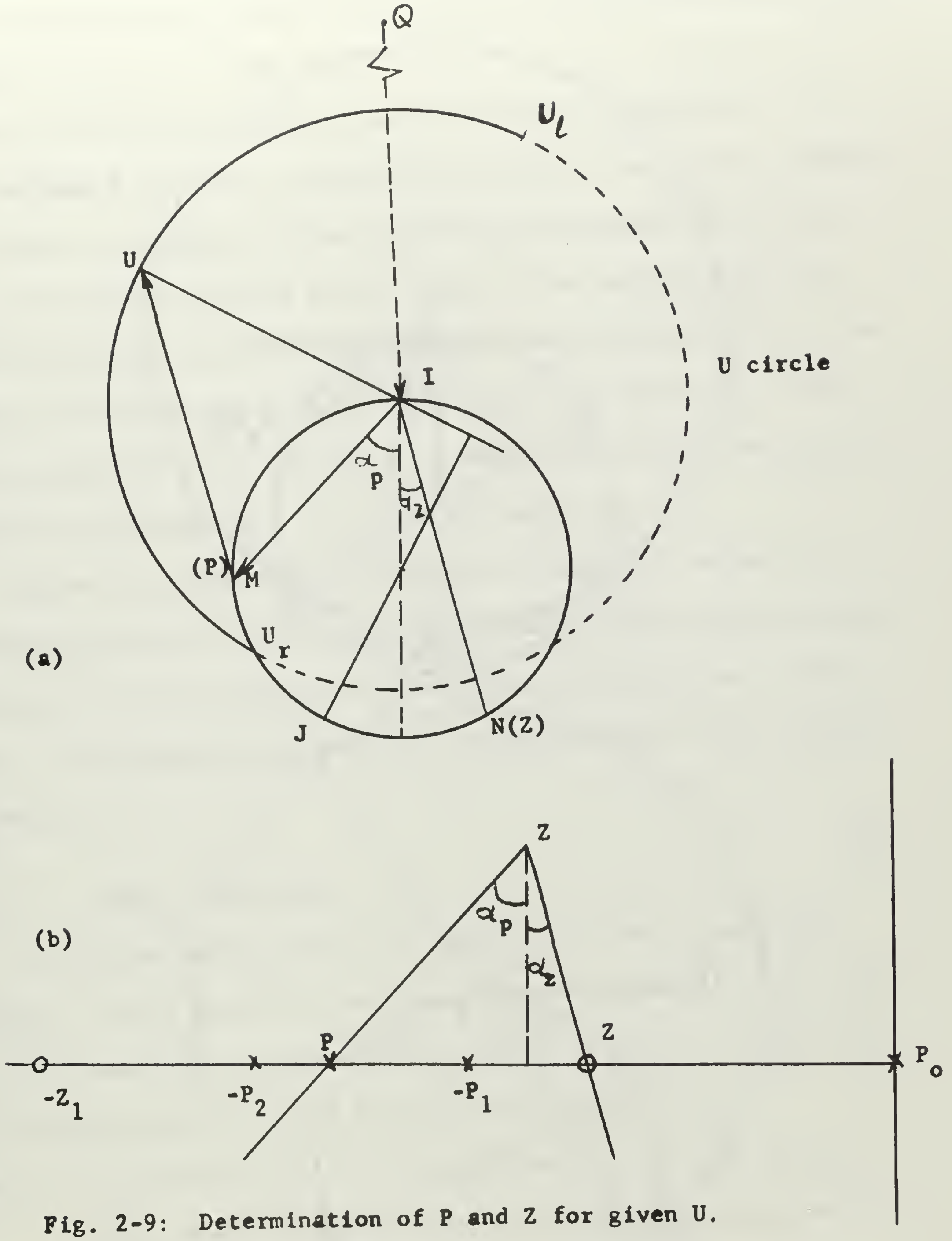


Fig. 2-9: Determination of P and Z for given U.

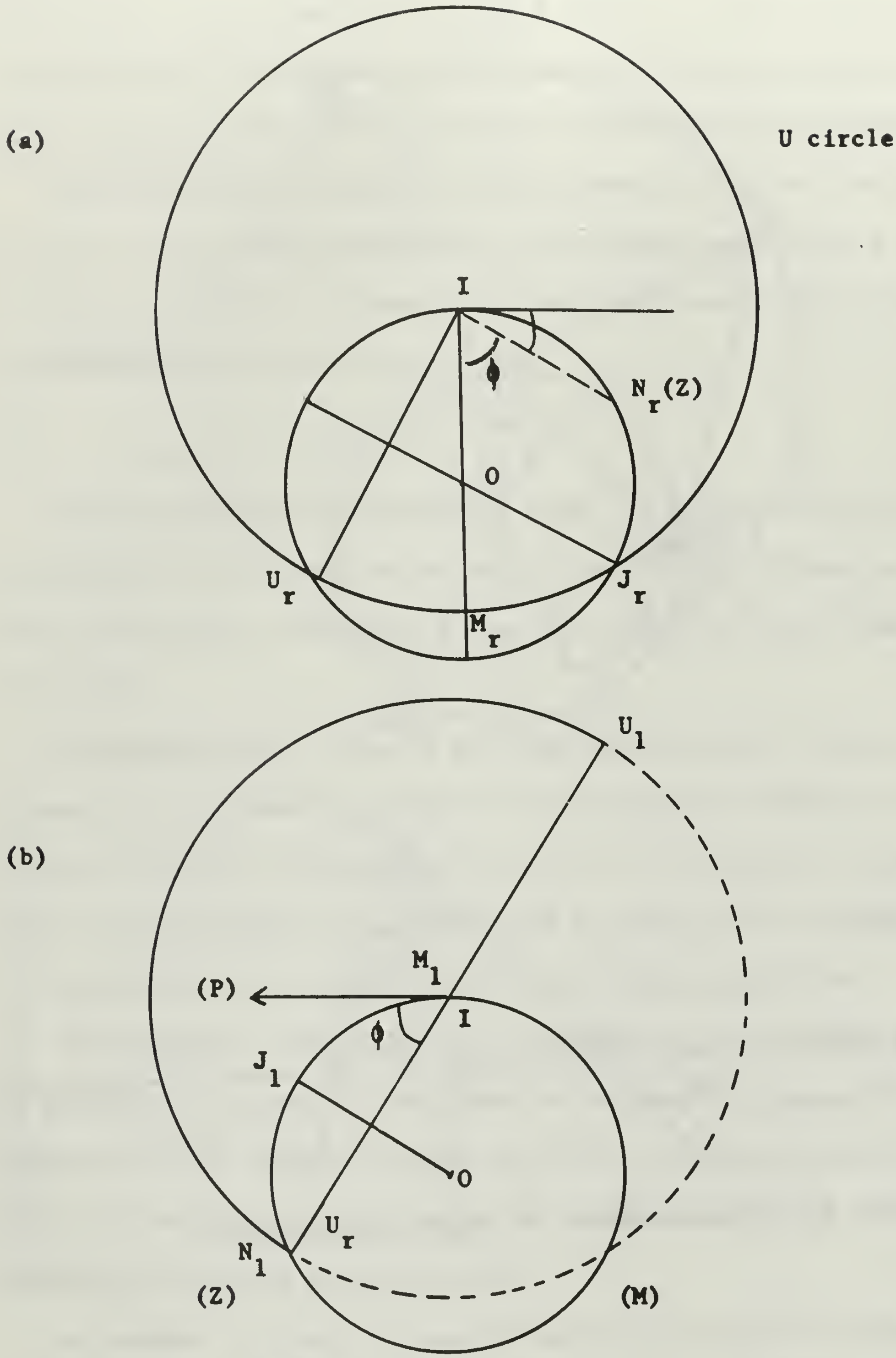


Fig. 2-10: Determination of the limit points of geometric locus of U.

2-9b P is obtained by drawing QP parallel to IM and Z is obtained by drawing QZ parallel to IN.

Now the problem recurs of determining the limit points of the U locus, using the same construction procedure. The extreme right limit point corresponds to the case where one compensator's singularity (here, the zero) is at the origin. The extreme left limit point corresponds to the case where the other singularity is at $-\infty$. Figure 2-10a shows how to obtain the right limit point, U_r. Drawing IN_r parallel to QO, because Z is now at O and take, arc $N_rJ_r = \phi$. Drawing IU_r perpendicular to OJ_r, then results in U_r.

Figure 2-10b shows how to obtain the left limit point, U_l. Draw IM_l parallel to QP, which is horizontal since P is at $-\infty$, and take arc $M_lJ_l = \phi$. Drawing IU_l perpendicular to OJ_l, then results in U_l. The geometric locus of U is the part of (U) circle between U_l and U_r.

II-13 - Design Techniques

In the preceeding section it has been derived a method for finding P and Z, given the desired location of point U on its locus, i.e. the magnitude and phase of QU, the unity vector. This method is restated below in a step by step form, Figure 2-8a.

Step 1 - Considering Q as if it already were a point on the root locus, draw the vector diagram at Q for the uncompensated system and obtain QI, the "uncompensated" unity vector.

Step 2 - Draw the circle diagram, composed of the (M) circle, of radius $R = \frac{1}{2d}$ and whose uppermost point is I, and the (U) circle centered at I and of radius $r = \frac{1}{d} \sin \phi$. Fix the

limits of the locus of U on the (U) circle.

Step 3 - Given QU as the desired unity vector for the compensated system, draw OJ penperpendicular to IU which cuts (M) circle at J. Draw angles JON = ϕ. Then Z and P are determined by drawing QZ parallel to IN and QP parallel to IM.

In this section, it will be seen how the "U" vector (i.e. the location of U on its locus) is selected to satisfy a particular condition.

(a) Design for Minimum Root-Sensitivity.

Since the U-vector is the scale used to measure the individual sensitivity vectors, the larger the scale, the smaller the magnitude of the sensitivity measures. Thus one possibility is to design the compensation for maximum magnitude of the U vector, that is minimum root sensitivity to open loop singularities. Figure 2-8a shows that maximum magnitude of QU is obtained by placing point U near the lower most part of the (U) circle or more exactly on the extension of QI. If such point is not within the locus of U, then it can not be a location for U and one must select the lowest point which is on the locus. In Figure 2-9a, this is point U_r. Thus QU_r is the selected unity vector, and with this given, one can proceed to the 3 step procedure outlined in the beginning of this section. With such a design $S_{P_i}^{s_j}$ and $S_{Z_i}^{s_j}$ are all minimum, for all i's.

(b) Design for Constant Damping When K Varies.

Another practical problem is to compensate a system in such a way that when gain K varies about its nominal value, the dynamic response of the system does not change. This calls for a constant ζ, i.e., a root locus that remains tangent to the radial line OQ at the

neighborhood of Q, Figure 2-8a. How can Z and P be found to obtain such a root locus? It is now shown that this can be done by merely selecting point U so that the unity vector QU goes through the origin, O, of the S-plane. In other words choose U so that Q, O and U are in line. If the locus of U does not permit such a choice, this means it is not possible to obtain a constant ξ about Q for the given system. One can then choose the best solution available, by taking the "U" location that is closest to a straight line with QO.

The above statement can be proven very simply if one recalls Equation (2-44) that

$$S_{P_o}^{s_j} = \frac{S_K^{s_j}}{s_j} \tag{2-63}$$

The specification here is to force $S_K^{s_j}$ to have the same direction as OQ. Thus making s_j move on a radial line when K varies. This means $\angle S_K^{s_j}$ must be equal to the phase of QO which is also phase of s_j. But from Equation (2-63)

$$\angle S_{P_o}^{s_j} = \angle S_K^{s_j} - \angle s_j = 0 \tag{2-64}$$

The phase of $S_{P_o}^{s_j}$ must be zero, this means that the sensitivity vector $S_{P_o}^{s_j}$ must be on the U-vector or conversely, the U vector must pass through P_o at the origin of the S-plane.

A faster way to prove this is that we show that $S_K^{s_j}$ always lies on the U vector. In order to keep ξ constant, $S_K^{s_j}$ must be radial, thus U vector must be radial.

II-14 - Limiting Behavior and Special Cases.

The magnitudes of gain sensitivities can cover the entire range of values from minus to plus infinity. Yet, intuitive notion of "sensitivity" as a general concept in closed loop systems makes part

of this range unreasonable. One part of the problem is a direct consequence of the sensitivity definition, while another is associated with its first order approximation nature. A better understanding of both facets can be gained by an examination of limiting cases.

In general, closed-loop poles depart from open-loop poles for low values of gain, and proceed to either open loop zeros or unbounded values as the open loop gain becomes very large. The gain sensitivity as given by Equation (2-13) could be written in the form

$$S_K^{s_j} = \frac{1}{\sum_{i=1}^{n} \frac{1}{Z_i - s_j} + \sum_{i=1}^{m+n} \frac{1}{s_j - P_i}} \tag{2-65}$$

As K approaches zero, the closed loop root $(-s_j)$ approaches the open loop pole from which it derives, i.e. $s_j \rightarrow P_i$. Then the term, the above relation reduces to

$$\left. S_K^{s_j} \right]_{K\rightarrow 0} \longrightarrow \frac{1}{\left(\frac{1}{s_j - P_i}\right)} \longrightarrow 0 \tag{2-66}$$

Similarly as K becomes very large, n of the closed-loop poles approach open loop zeros. If the jth closed-loop pole is one of these, and it approaches the ith open loop zero so that

$$s_j \longrightarrow Z_i$$

Then,

$$\left. S_K^{s_j} \right]_{K\rightarrow \infty} \longrightarrow \frac{1}{\left(\frac{1}{Z_i - s_j}\right)} \longrightarrow 0 \tag{2-67}$$

Finally, m of the closed-loop poles have no zeros to go to, and hence become very large relative to P_i and Z_i. The sensitivity for these poles is

$$\left. S_K^{s_j} \right]_{s_j \gg Z_i, P_i} \longrightarrow \frac{1}{\sum_{i=1}^{m} \frac{1}{s_j}} = \frac{s_j}{m} \tag{2-68}$$

Here m = (order of denominator) - (order of numerator). When the gain is sufficiently large for the open-loop zero db line to intersect the high frequency asymptote on Bode plot, the open loop transfer function is approximately

$$L(s) = \frac{K}{S^m}$$

where m = (order of denominator) - (order of numerator). Then for the points on root locus

$$1 + L(s)\Big|_{s = -s_j} = 0$$

or

$$1 + \frac{K}{(-s_j)^m} = 0$$

or

$$s_j = - \sqrt[m]{-K} \tag{2-69}$$

Thus, the sensitivity of the unbounded pole from (2-68) will be

$$S_K^{s_j}\Bigg]_{s_j \gg Z_i, P_i} \longrightarrow - \frac{\sqrt[m]{-K}}{m} \tag{2-70}$$

Equation (2-70) indicates that sensitivity increases as the m root of K, as gain is increased, although for the finite gains the sensitivity is always finite. Referring to Figure 2-11, the above special cases can be simplfied as:

For poles going towards "O.L. zeros" $\begin{cases} \text{when } K \to \infty & S_K^{s_j} \to 0 \\ \text{when } K \to 0 & S_K^{s_j} \to 0 \end{cases}$

For poles going towards infinity $\begin{cases} S_K^{s_j} = \dfrac{s_j}{m} & \text{in general} \\ S_K^{s_j} = -\dfrac{\sqrt[m]{-K}}{m} & \text{when } L = \dfrac{K}{s^m} \end{cases}$

Another circumstance in which the sensitivity can become very large is revealed by Equation (2-45), and that happens when Q_j (the

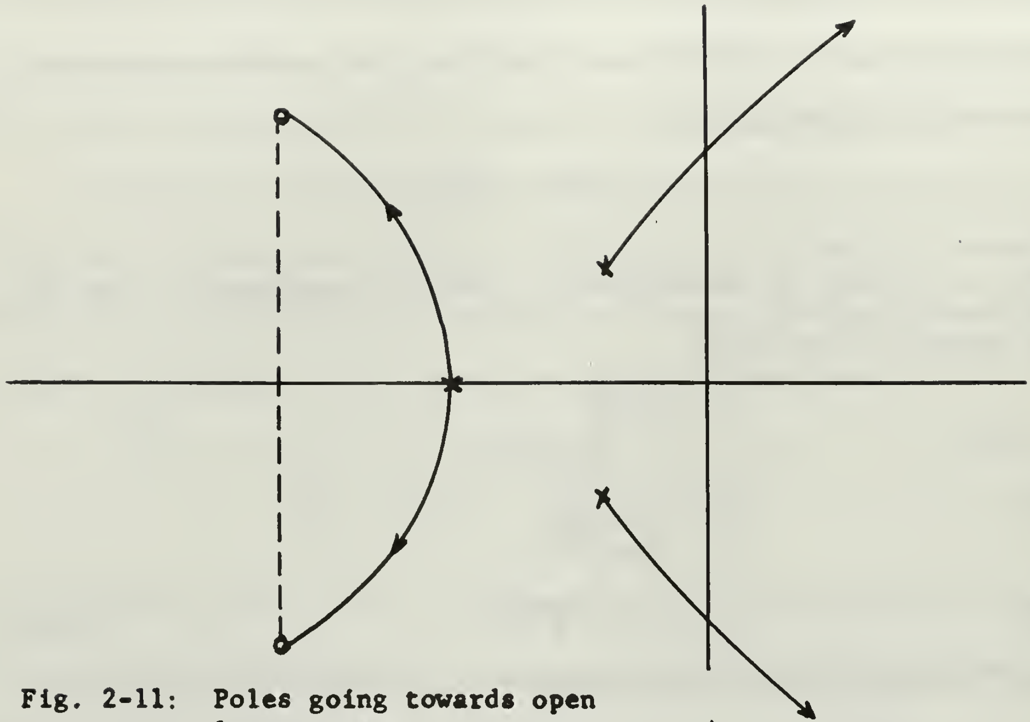


Fig. 2-11: Poles going towards open loop zeros.

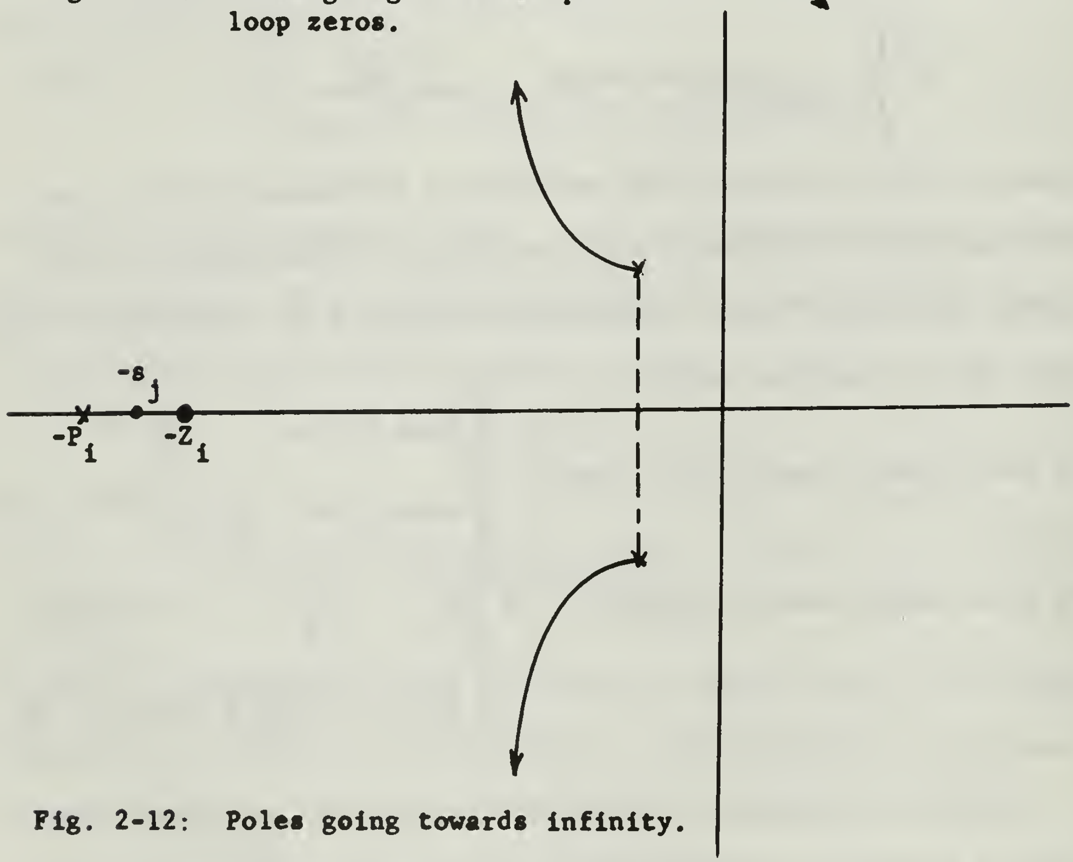


Fig. 2-12: Poles going towards infinity.

residue evaluated at pole $-s_j$) is not finite. This is to be expected since the sensitivity factors defined thus far have not considered multiple-order, closed-loop roots. As far as the gain is finite, an infinite gain sensitivity always indicates multiple-order closed-loop poles.

A special situation of considerable interest can occur when a closed loop root lies between an open-loop pole and zero which are much closer to each other than to all other open-loop poles and zeros. This is the so called dipole case as in Figure 2-12. The sensitivity for the bounded closed-loop pole will be, approximately

$$S_K^{s_j} = \frac{1}{\dfrac{1}{Z_i - s_j} + \dfrac{1}{s_j - P_i}} = \frac{(Z_i - s_j)(s_j - P_i)}{Z_i - P_i} \tag{2-71}$$

Differentiating equation (2-71) to find values of s_j which make, sensitivity maximum, reveals that maximum value of $S_K^{s_j}$ will occur when $s_j = \frac{Z_i + P_i}{2}$ for which $S_K^{s_j}$ becomes

$$\left. S_K^{s_j} \right]_{max} = \frac{1}{4} (Z_i - P_i) \tag{2-72}$$

II-15 - Sensitivity at Irregular Points

Expanding the gain $K = K(s)$ in series to include the higher order terms we have

$$K = K_o + \frac{dK}{ds} \Delta s + \frac{1}{2} \frac{d^2K}{ds^2} \Delta s^2 + \cdots \tag{2-73}$$

and then assume the first n-1 derivatives vanish. This means then,

$$\Delta K = K - K_o = \frac{1}{n!} \frac{d^nK}{ds^n} \Delta s^n$$

or

$$\Delta s = \sqrt[n]{\frac{n!\ \Delta K}{\dfrac{d^nK}{ds^n}}} = \sqrt[n]{\frac{n!}{\dfrac{d^nK}{ds^n}}}\ \sqrt[n]{\Delta K} \qquad (2\text{-}74)$$

Since $\dfrac{\Delta s}{\sqrt[n]{\Delta K}}$ is finite it is suggested to be substituted for $S_K^{s_j}$ at irregular points.

For any $\Delta K > 0$, ds has n values separated by $\frac{2\pi}{n}$ which represents the outgoing branches in Figure 2-13. If $\Delta K < 0$, n branches with the same separation as before are again obtained, but they are displaced from the previous set by π/n, that is the two set intersect. This could be stated as: The incoming branches, which at their junction represent an nth order closed-loop pole, are evenly spaced and separated from each other by $2\pi/n$. The outgoing branches are also separated from each other by $2\pi/n$ and are midway between the incoming branches.

II-16 - Sensitivity Functions for Alternate Transfer Function Forms

The transfer function could be written either in root locus or Bode form. For Figure 2-6, open-loop transfer function can be written as

$$G(s) = \frac{C(s)}{E(s)} = K\ \frac{s^n + a_1 s^{n-1} + \cdots + a_n}{s^{n+m} + b_1 s^{m+n-1} + \cdots b_{m+n}}$$

$$= K\ \frac{\sum_{i=0}^{n} a_i s^{n-i}}{\sum_{i=0}^{m+n} b_i s^{m+n-i}} = K\ \frac{\prod_{i=1}^{n} (s + Z_i)}{\prod_{i=1}^{m+n} (s + P_i)} \qquad (2\text{-}75a)$$

$$= \frac{K' \prod_{i=1}^{m} \left(\frac{s}{Z_i} + 1\right)}{\prod_{i=1}^{m+n} \left(\frac{s}{P_i} + 1\right)} \qquad (2\text{-}75b)$$

The Equations (2-75a) and (2-75b) represent the root locus and Bode form respectively.

The equations for the gain sensitivity are the same whether open-loop transfer function is in root locus or Bode form, i.e.

$$S_K^{s_j} = S_{K'}^{s_j} \qquad (2\text{-}76)$$

This is because in comparing (2-75a) and (2-75b) it is obvious that

$$K' = K \frac{\prod_i Z_i}{\prod_i P_i} = K \times \text{constant}$$

and then,
$$S_K^{s_j} = \frac{\partial s_j}{\frac{\partial K}{K}} = \frac{\partial s_j}{\frac{\partial K'}{K'}}$$

It is however necessary to modify the open loop pole and zero sensitivities for terms which are written in Bode form. For poles and zeros which appear in open loop transfer function in Bode form, i.e. $\left[\left(\frac{S}{P_i}\right) + 1\right]$ or $\left[\left(\frac{S}{Z_i}\right) + 1\right]$, the sensitivities are

$$S_{Z_i}^{s_j} = \frac{s_j\, S_K^{s_j}}{Z_i(Z_i - s_j)} \qquad (2\text{-}77)$$

and

$$S_{P_i}^{s_j} = \frac{s_j\, S_K^{s_j}}{P_i(s_j - P_i)} \qquad (2\text{-}78)$$

Frequently open-loop zeros and poles will occur as complex conjugate pairs, and variations in the system will change both zeros or poles. For example, consider a complex pair of zeros, Z_1 and Z_2, which are defined by their frequency, ω, and damping ratio ζ, i.e.

$$\begin{aligned} Z_1 &= \zeta\omega + j\omega\sqrt{1 - \zeta^2} \\ Z_2 &= \zeta\omega - j\omega\sqrt{1 - \zeta^2} \end{aligned} \qquad (2\text{-}79)$$

For this situation it is useful to define frequency and damping ratio sensitivities as

$$S_{\omega}^{s_j} = \frac{\partial s_j}{\partial \omega} = \frac{\partial s_j}{\partial Z_1}\frac{dZ_1}{d\omega} + \frac{\partial s_j}{\partial Z_2}\frac{dZ_2}{d\omega}$$

$$S_{\zeta}^{s_j} = \frac{\partial s_j}{\partial \zeta} = \frac{\partial s_j}{\partial Z_1}\frac{dZ_1}{d\zeta} + \frac{\partial s_j}{\partial Z_2}\frac{dZ_2}{d\zeta} \qquad (2\text{-}80)$$

Where $\frac{\partial s_j}{\partial Z1}$ and $\frac{\partial s_j}{\partial Z_2}$ are given according to (2-42) or (2-43) (In case of poles). Components as $\frac{dZ_1}{d\omega}$, $\frac{dZ_1}{d\zeta}$, $\frac{dZ_2}{d\omega}$, and $\frac{dZ_2}{d\zeta}$ can easily be evaluated from (2-79). Then when the complex pair of zeros appear in open-loop transfer functions in root locus form, i.e. $(s^2+2\zeta\omega s+\omega^2)$, the frequency and damping ratio sensitivities for a complex pair of poles or zeros are

$$S_{\omega}^{s_j} = \frac{\pm 2(\omega - \zeta s_j) S_K^{s_j}}{s_j^2 - 2\zeta\omega s_j + \omega^2}, \quad S^{s_j} = \frac{\mp 2\omega s_j S_K^{s_j}}{s_j^2 - 2\zeta\omega s_j + \quad ^2} \qquad (2\text{-}81)$$

Where the upper sign is to be used for zeros and the lower one for poles. If the term is written in Bode form, i.e. $\left[\frac{s^2}{\omega^2} + \frac{2\zeta s}{\omega} + 1\right]$ the sensitivities are

$$S_{\omega}^{s_j} = \frac{\mp 2s_j}{\omega} \quad \frac{(\zeta\omega - s_j) S_K^{s_j}}{s_j^2 - 2\zeta\omega s_j + \omega^2}$$

$$S_{\zeta}^{s_j} = \frac{\mp 2 s_j \omega S_K^{s_j}}{s_j^2 - 2\zeta\omega s_j + \omega^2} \qquad (2\text{-}82)$$

For some cases it may be more convenient to define a complex pair in terms of their real and imaginary parts, i.e.

$$Z_1 = a + jb \qquad Z_2 = a - jb$$

In this case with the terms in root locus form $[s^2 + 2as + (a^2 + b^2)]$, the sensitivities are

$$S_a^{s_j} = \frac{\partial s_j}{\partial a} = \frac{\partial s_j}{\partial Z_1} \bullet \frac{dZ_1}{da} + \frac{\partial s_j}{\partial Z_2} \bullet \frac{dZ_2}{da}$$

$$= \frac{S_K^{s_j}}{Z_1 - s_j}(1) + \frac{S_K^{s_j}}{Z_2 - s_j}(1)$$

$$= S_K^{s_j}\left[\frac{1}{(a + jb - s_j)} + \frac{1}{(a - jb - s_j)}\right]$$

$$S_a^{s_j} = \frac{2(a - s_j)\, S_K^{s_j}}{s_j^2 - 2as_j + a^2 + b^2}$$

and in a more general form as in case of Equation (2-81) it could be written as

$$S_a^{s_j} = \frac{\partial s_j}{\partial a} = \frac{\pm 2(a - s_j)\, S_K^{s_j}}{s_j^2 - 2as_j + a^2 + b^2} \qquad (2\text{-}83)$$

$$S_b^{s_j} = \frac{\partial s_j}{\partial b} = \frac{\pm\, 2b\, S_K^{s_j}}{s_j^2 - 2as_j + a^2 + b^2}$$

where the upper sign is to be used for zeros and the lower one for poles. If the term is in Bode form $\left([s^2/(a^2 + b^2)] + [2as/(a^2 + b^2)] + 1\right)$, the sensitivities are

$$S_a^{s_j} = \left[\frac{\pm\, 2s_j}{a^2 + b^2}\right]\left[\frac{(a^2 - b^2 - as_j)\, S_K^{s_j}}{s_j^2 - 2as_j + a^2 + b^2}\right] \qquad (2\text{-}84)$$

$$S_b^{s_j} = \left[\frac{\pm\, 2s_j b}{a^2 + b^2}\right]\left[\frac{(2a - s_j)\; S_K^{s_j}}{s_j^2 - 2as_j + a^2 + b^2}\right]$$

II-17 - Example

In Figure 2-14, suppose the gain K, is given as

$$K = -\frac{s^2 + 1}{s} = -s - \frac{1}{s}$$

Find radius of curvature, and pole sensitivity.

We can write
$$\frac{dK}{ds} = -1 + \frac{1}{s^2}$$

$$\frac{d^2K}{ds^2} = -\frac{2}{s^3}$$

$$\frac{d^3K}{ds^3} = \frac{6}{s^4}$$

Double points occur at $\frac{dK}{ds} = 0$ or $S = \pm 1$. Substituting $S = 1\, e^{j\phi}$,

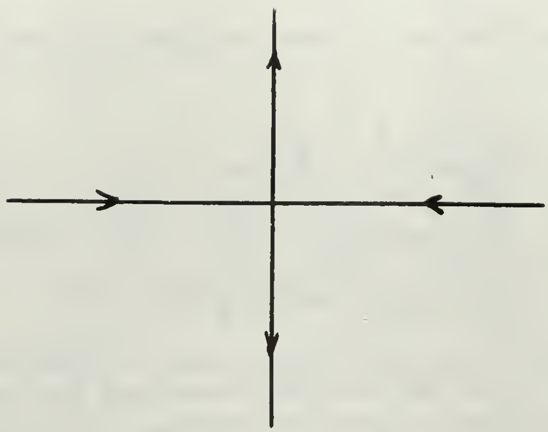


Fig. 2-13: Outgoing Branches (n = 2)

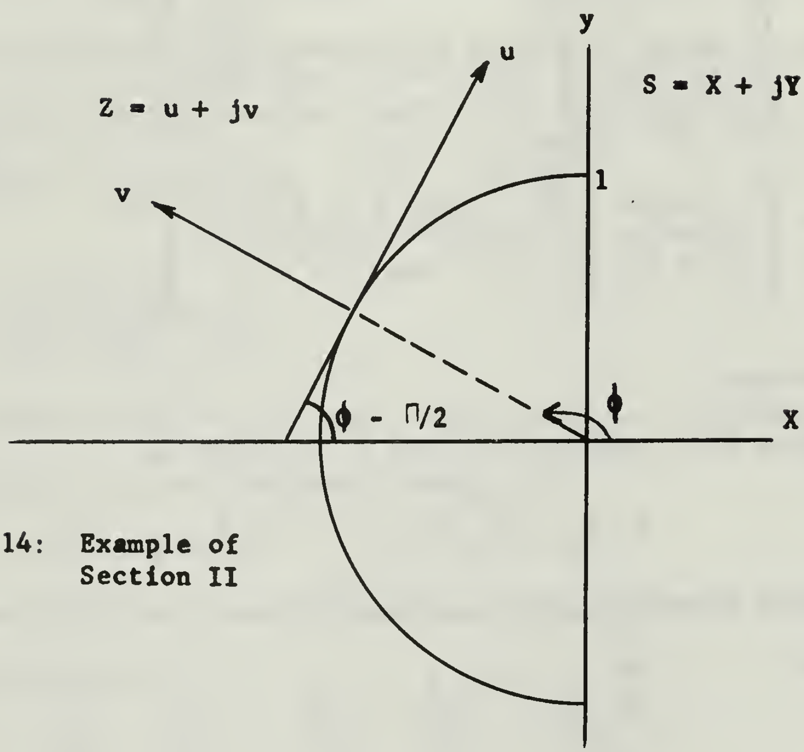


Fig. 2-14: Example of Section II

we get

$$\frac{dK}{ds} = -1 + \frac{1}{e^{2j\phi}}$$

$$= -1 + e^{-2j\phi}$$

$$= 2je^{-j\phi} \; \frac{e^{j\phi} - e^{-j\phi}}{2j}$$

$$= -2je^{-j\phi} \sin \phi$$

$$= 2 \sin \phi \; e^{-j(\phi + \frac{\pi}{2})}$$

On circular portion of the root locus,

$$\frac{dy}{dx} = - \frac{\text{Im} \frac{dK}{ds}}{\text{Re} \frac{dK}{ds}} = - \cot \phi = \tan \left(\phi - \frac{\pi}{2}\right)$$

The pole sensitivity is

$$S_K^{s_j} = K \frac{ds}{dK} = - \frac{s^2 + 1}{s} \bullet \frac{s^2}{1 - s^2}$$

$$= s \; \frac{s^2 + 1}{s^2 - 1}$$

$$= s \; \frac{s + \frac{1}{s}}{s - \frac{1}{s}}$$

On the circular portion

$$S_K^{s_j} = e^{j\phi} \; \frac{e^{j\phi} + e^{-j\phi}}{e^{j\phi} - e^{-j\phi}} = -je^{j\phi} \cot \phi$$

The pole sensitivity at $s = -1$ (a double point) is infinite and hence the suggested substitute, i.e. $\frac{\Delta s}{\sqrt[n]{\Delta K}}$ is evaluated. Here $n = 2$ and then

$$\frac{\Delta s}{\sqrt[n]{\Delta K}} = \sqrt[n]{\frac{n!}{\frac{d^n K}{ds^n}}}$$

$$= \sqrt{- \frac{2!}{\frac{2}{s^3}}} = \pm 1 \text{ evaluated at } s = -1$$

III

SENSITIVITY AND LARGE PARAMETER VARIATIONS

III-1 - Discussion

This section deals with the most practical aspect of the sensitivity and is quite utilized in design consideration of the feedback control systems. It discusses sensitity in two particular cases one with leakage transmission, and the other case when leakage transmission is zero. It relates the sensitivity of the system to return difference and null return difference.

In previous sections it has been discussed that in most cases we are facing the variation in the values of an element in an engineering system. This element could be a vacuum tube, transistor or sometimes could just be a passive element. It is possible by means of feedback around the troublesome element, to achieve the desired reduction in sensitivity. Let the troublesome parameter be k, as in Appendix 1. We relate k to the controlled source S and control variable C, i.e. $S = kC$ and,

$$T = t_{oi} + \frac{kt_{ci}t_{os}}{1 - kt_{cs}} \tag{3-1}$$

with the fundamental signal flow graph as shown in Appendix 1.

We defined the classical or system sensitivity as,

$$S_k^T = \frac{\partial \ln T}{\partial \ln k} = \frac{\partial T/T}{\partial k/k} \tag{3-2}$$

so S_k^T is the relative change in T divided by the relative change in k for infinitesimally small changes in T and k only.

<u>Case 1:</u>

Consider the case where $t_{oi} \ll T$, i.e. leakage transmission is negligible. This is a situation in many feedback systems. By straight

forward differentiation of Equation (3-1) we get

$$S_k^T = \frac{\partial \ln T}{\partial \ln k} = \frac{\partial T/T}{\partial k/k} = \frac{k}{T}\frac{\partial T}{\partial k}$$

$$= \frac{k}{\dfrac{kt_{ci}t_{os}}{1 - kt_{cs}}}\left[\frac{t_{ci}t_{os}(1 - kt_{cs}) + t_{cs}(kt_{ci}t_{os})}{(1 - kt_{cs})^2}\right]$$

$$= \frac{k(1 - kt_{cs})}{kt_{ci}t_{os}}\left[\frac{t_{ci}t_{os}}{(1 - kt_{cs})^2}\right]$$

and from that

$$S_k^T = \frac{1}{1 - kt_{cs}} = \frac{1}{F_k} \quad \text{for } t_{oi} = 0 \qquad (3\text{-}3)$$

where F_k is the return difference for reference k. It is clear now why it is insisted that all t_{ij}'s in the fundamental feedback equation should be independent of k. To have a small sensitivity then $\left|F_k\right|$ should be large, i.e. $(-kt_{cs})$ which is usually called the loop gain should be a large number.

Since t_{cs} and possibly k is a function of frequency, so is S_k^T, and one must consider the frequency range over which a small S_k^T is desired. It is interesting to note that as the magnitude of the loop transmission determines the sensitivity, its phase also determines the stability of the system.

Case 2:

When the leakage transmission $t_{oi} \neq 0$, then by straight forward differentiation as in Case 1 we get

$$\frac{\partial T}{\partial k} = \frac{t_{ci}t_{os}}{(1 - kt_{cs})^2}$$

and $$S_k^T = \frac{k}{T}\frac{\partial T}{\partial k} = \frac{kt_{ci}t_{os}}{T(1 - kt_{cs})^2}$$

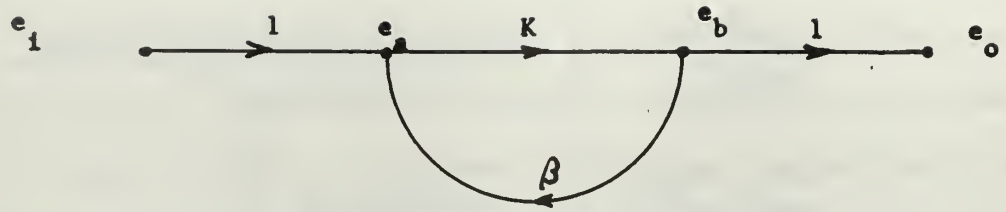


Fig. 3-1: Signal Flow Graph for Example (1).

Fig. 3-2: Signal Flow Graph for Example (2).

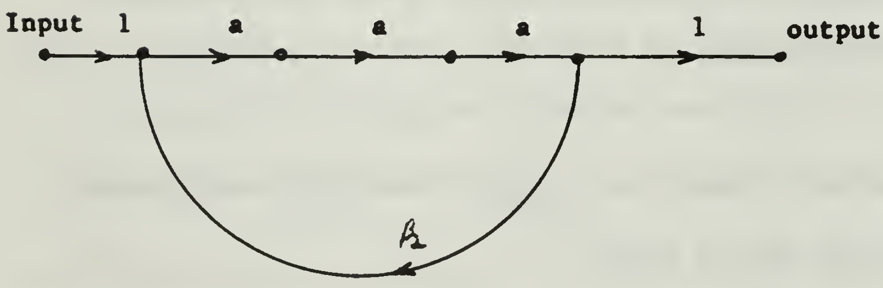


Fig. 3-3: Signal Flow Graph for Example (3).

This could be written in the form

$$S_k^T = \frac{1}{(1 - kt_{cs})} \left[\frac{kt_{ci}t_{os}}{T(1 - kt_{cs})} \right]$$

and since from Equation (3-1)

$$T - t_{oi} = \frac{kt_{ci}t_{os}}{1 - kt_{cs}}$$

then
$$S_k^T = \frac{1}{1 - kt_{cs}} \left[1 - \frac{t_{oi}}{T} \right]$$

$$= \frac{1}{F_k} \left[1 - \frac{t_{oi}}{T} \right] \tag{3-4}$$

S_k^T can also be written in terms of return difference and null return difference. From Appendix 1, Equation (A3-3), we have

$$\frac{t_{oi}}{T} = \frac{F_k}{F_k'}$$

then
$$S_k^T = \frac{1}{F_k} - \frac{1}{F_k'} \tag{3-5}$$

where F_k is the return difference with reference to k and F_k' is the null return difference also with reference to k. Equation (3-4) suggests a practical mean to measure the sensitivity. It is noted that in case $t_{oi} = 0$, Equation (3-4) reduces to Equation (3-2). The applications of Equation (3-4) are illustrated by a number of simple examples.

Example - 1

In the single-loop system Figure 3-1, if the sensitivity desired is with respect to the forward transmission, the leakage transmission is zero, because with K = 0 there is no transmission from input to output. The sensitivity in this case is the reciprocal of the return difference, i.e.

$$S_K^T = \frac{1}{F_K} = \frac{1}{1 - \beta K} \qquad (3\text{-}6)$$

Example - 2

In the same single-loop system, if the sensitivity of interest is with respect to the feedback transmission, β, the leakage transmission will then be K, and Equation (3-5) yields

$$S_\beta^T = \frac{1}{1 - \beta K}\left(1 - \frac{K}{K/(1 - \beta K)}\right) = \frac{\beta K}{1 - \beta K} \qquad (3\text{-}7)$$

In this example it is noted that when $\beta = 0$, i.e., there is no feedback around k, then the leakage transmission $t_{oi} = 1 \times K \times 1 = K$. Comparison of Equations (3-6) and (3-7) demonstrates the well known fact that a value of K large with respect to unity results insensitive to changes in K, but with the closed-loop gain essentially equal to $\frac{1}{\beta}$ (when K is very large) the sensitivity of T with respect to β approximates the value of $-1 = 1 \,\underline{/18^{o}}$.

Example - 3

Figures 3-2 and 3-3 represent alternative designs for a feedback amplifier. Three amplifier stages are available, each with a gain a, and the overall gain to be realized is specified (less than a^3). Negative feedback is to be introduced to reduce the sensitivity of the overall gain to variations in supply voltage. To a first approximation this voltage variation can be considered equivalent to a variation in each value of a. The question to be answered is which configuration gives a lower value of sensitivity of the overall gain with respect to a? For Figure 3-2, the overall gain T_1 is

$$T_1 = \left[\frac{a}{1 - \beta_1 a}\right]^3$$

The sensitivity of T_1 with respect to a is three times the sensitivity

of an individual stage with feedback, or

$$S_a^{T_1} = \frac{3}{1 - \beta_1 a} \tag{3-8}$$

For the second configuration, with a single overall feedback path, the overall gain T_2 is

$$T_2 = \frac{a^3}{1 - \beta_2 a^3}$$

and

$$S_a^{T_2} = \frac{3}{1 - \beta_2 a^3} \tag{3-9}$$

The two sensitivites (3-8) and (3-9) are to be compared on the basis of the equality of T_1 and T^2. This equality establishes the relation

$$(1 - \beta_1 a)^3 = 1 - \beta_2 a^3 \tag{3-10}$$

substituting (3-10) into (3-9) we get

$$S_a^{T_2} = \frac{3}{(1 - \beta_1 a)^3}$$

Thus, the single overall feedback path results in a system with less sensitivity, to changes in a.

III-2 - Large Parameter Changes.

A serious shortcoming of Equations (3-1), (3-2), and (3-3) is that they apply only for infinitisimally small changes of k. There is therefore uncertainty as to their applicability for moderate or large changes in k. It is therefore found useful to use a new definition, defined as follows:

Let T_o, k_o represent the nominal or original design value of the system transfer function and of the element under consideration respectively, and let T_f, k_f be the corresponding (final) values at the new value of k. Thus,

$$T_f = T_o + \Delta T \qquad k_f = k_o + \Delta k \tag{3-11}$$

The new sensitivity function is defined as

$$S_k^T \stackrel{D}{=} \frac{\Delta T/T_f}{\Delta k/k_f} \tag{3-12}$$

<u>Case 1</u>:

When it is assumed that $t_{oi} = 0$, then from Equation (3-1),

$$T_o = \frac{k_o t_{ci} t_{os}}{1 - k_o t_{cs}} \tag{3-13}$$

$$T_f = \frac{k_f t_{ci} t_{os}}{1 - k_f t_{cs}} \tag{3-14}$$

$$S_k^T = \frac{\Delta T/T_f}{\Delta k/k_f} = \frac{\dfrac{T_f - T_o}{T_f}}{\dfrac{k_f - k_o}{k_f}} \tag{3-15}$$

Substituting (3-13) and (3-14) into (3-15) we get

$$S_k^T = \frac{1 - \dfrac{T_o}{T_f}}{\dfrac{k_f - k_o}{k_f}} = \frac{1 - \dfrac{k_o(1 - k_f t_{cs})}{k_f(1 - k_o t_{cs})}}{\dfrac{k_f - k_o}{k_f}}$$

$$= \frac{1}{1 - k_o t_{cs}} = \frac{1}{F_{k_o}} = \frac{1}{1 + L_o} \tag{3-16}$$

where F_{k_o} is the initial return difference and L_o is also the initial loop transmission. It is thus shown that when the leakage transmission t_{oi} is zero, the classical and new definition have the same value, although they are defined differently. With the new definition there is no uncertainty as to the effect of large parameter variation. Equation (3-16) is exact only if t_{oi} is zero, but it may be used whenever $t_{oi} \ll T_f, T_o$. Fortunately t_{oi} is zero for a very large class of feedback amplifiers. From Equations (3-11) and (3-16) we have

$$\frac{T_f - T_o}{T_f} = 1 - \frac{T_o}{T_f} = \frac{\Delta T}{T_f} = S_k^T \frac{\Delta k}{k_f}$$

$$= \frac{1}{1 - k_o t_{cs}} \left(\frac{k_f - k_o}{k_f} \right)$$

$$= \frac{1}{1 - k_o t_{cs}} \left(1 - \frac{k_o}{k_f} \right)$$

Therefore

$$\frac{T_o}{T_f} = 1 - (1 - \frac{k_o}{k_f}) \left(\frac{1}{1 - k_o t_{cs}} \right)$$

$$= \left(\frac{k_o}{k_f} - k_o t_{cs} \right) \left(\frac{1}{1 - k_o t_{cs}} \right)$$

and

$$\frac{T_o}{T_f} = \frac{\frac{k_o}{k_f} + L_o}{1 + L_o} \qquad (3\text{-}17)$$

where $L_o \overset{D}{=} -k_o t_{cs}$ is the original or the nominal loop transmission and Equation (3-17) relates the original and final values of the closed loop transfer function which is of very practical design application.

Case 2

In previous cases we assumed t_{oi} was negligibly small in comparison with T_o and T_f over the range of variation of k. This is not always true. Suppose in a system whose signal flow graph representation is that shown in Figure 3-4 parameter D varies substantially, and feedback is to be used to reduce the system sensitivity to the variations in D. If we choose D = k of the fundamental feedback equation, then t_{oi} may not be small. However the problem can be handled by method in Case 1, if we simply let k represent the entire element, i.e.

$$k = \frac{A(B + ED)}{1 - AED}$$

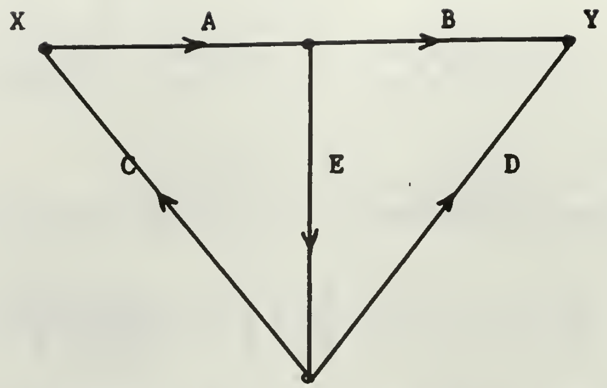


Fig. 3-4: A Signal Flow Graph with Leakage Transmission.

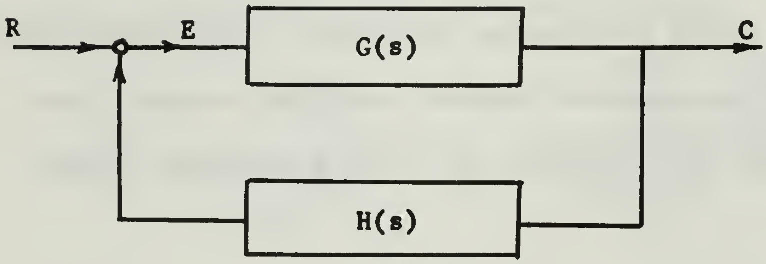


Fig. 3-5: Basic Feedback System.

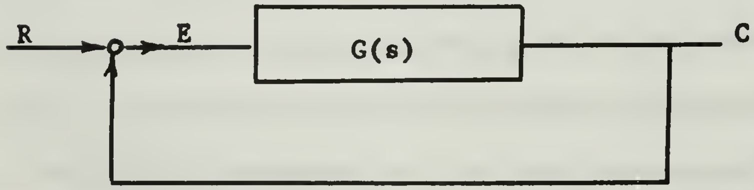


Fig. 3-6: Basic Unity Feedback System.

The element k is then incorporated into the overall system feedback configuration. The problem then reduces to Case 1.

In the above manner the majority of feedback problems can be put into zero leakage category. For those situations where a parallel path from the overall system input node to the output node (from I to 0) exists, the following results are obtained:

Since $$T = t_{oi} + kt_{ci}t_{os} / (1 - kt_{cs})$$

using Equation (3-11) and notation (3-12) we get

$$S_k^T = \frac{\Delta T/T_f}{\Delta k/k_f} = \frac{1}{(1 - k_o t_{cs})} \left(1 - \frac{t_{oi}}{T_f}\right)$$

or $$S_k^T = \left(\frac{1}{1 + L_o}\right) \left(1 - \frac{t_{oi}}{T_f}\right) \tag{3-18}$$

Equation (3-18) can also be written as

$$S_k^{(T-t_{oi})} = \frac{1}{1 + L_o} \tag{3-19}$$

and from (3-19) by using Equation (3-17) we get

$$\frac{T_o - t_{oi}}{T_f - t_{oi}} = \frac{(k_o/k_f) + L_o}{1 + L_o}$$

or $$\frac{T_f}{T_o} = \frac{(1 + L_o) - (t_{oi}/T_o)(1 - k_o/k_f)}{L_o + k_o/kf}$$

and $$\frac{\Delta T}{T_o} = \frac{T_f - T_o}{T_o} = \frac{(1 - t_{oi}/T_o)\,(1 - k_o/k_f)}{L_o + (k_o/k_f)} \tag{3-20}$$

III-3 - Further Investigation into the Meaning of Sensitivity Function.

In prev ious discussions we defined the system sensitivity as

$$S_K^T = \frac{1}{1 + L} \tag{3-21}$$

where L is defined as the open loop transfer function, and T is the closed loop transfer function. Let us now investigate the value

$$\frac{dT/T}{dL/L}$$

Referring to Figure 3-5 we have

$$T = \frac{G}{1 + GH} = \frac{1}{H}\frac{GH}{1 + GH}$$

But $\qquad L = GH \qquad \text{and} \qquad dL = HdG$

$$\frac{dT/T}{dL/L} = \frac{L}{T}\frac{dT}{dL} = \frac{L}{HT}\frac{dT}{dH}$$

$$= \frac{1}{1 + GH} = \frac{1}{1 + L}$$

Now by definition $\frac{dT/T}{dL/L} \overset{D}{=} S_O^C$ i.e., sensitivity of the closed-loop to open-loop perturbations, and comparing Equations (3-21) and (3-22) we get

$$S_K^T = S_O^C \qquad (3\text{-}23)$$

Also from relation

$$S_K^T = S_O^C = \frac{1}{1 + L}$$

we can think of the sensitivity functions as the error transfer function for a unity feedback system with forward elements GH, because

$$R - C = E$$

or

$$R - \frac{RGH}{1 + GH} = E$$

then

$$\frac{E}{R} = \frac{1}{1 + GH} = \frac{1}{1 + L}$$

In control system, the error transfer function is always desired small, ideally zero, and we may therefore observe that a small sensitivity function is desirable.

Since the sensitivity function is dependent upon frequency and is a complex number for any arbitrary frequency we must further inquire

into its meaning.

Let
$$L = \Gamma e^{j\phi}$$

then
$$S_o^c = S_K^T = \frac{1}{1 + \Gamma e^{j\phi}}$$

$$= \frac{1}{(1 + \Gamma\cos\phi) + j(\Gamma\sin\phi)}$$

or
$$S_o^c = S_K^T = \frac{1 + \Gamma\cos\phi}{1 + 2\Gamma\cos\phi + \Gamma^2} - j\,\frac{\Gamma\sin\phi}{1 + 2\Gamma\cos\phi + \Gamma^2} \tag{3-24}$$

or
$$S_o^c = S_K^T = \frac{1}{\sqrt{1 + 2\Gamma\cos\phi + \Gamma^2}}\, e^{-j\tan^{-1}\left[\frac{\Gamma\sin\phi}{1 + \Gamma\cos\phi}\right]} \tag{3-25}$$

It is desirable to interpret both of the above equations from a physical view point. We then proceed in this direction.

A - A Synthetic Interpretation of S_o^c

We define the following four quantities:

$$S_M^M \stackrel{D}{=} \frac{d|T|/|T|}{d|L|/|L|} = \frac{\text{per unit increment in } |T|}{\text{per unit increment in } |L|} \tag{3-26}$$

$$S_A^A \stackrel{D}{=} \frac{d\,\angle T}{d\,\angle L} = \frac{\text{increment in } \angle T}{\text{increment in } \angle L} \tag{3-27}$$

$$S_A^M \stackrel{D}{=} \frac{d|T|/|T|}{d\,\angle L} = \frac{\text{per unit increment in } |T|}{\text{increment in } \angle L} \tag{3-28}$$

$$S_M^A \stackrel{D}{=} \frac{d\,\angle T}{d|L|/|L|} = \frac{\text{increment in } \angle T}{\text{per unit increment in } |L|} \tag{3-29}$$

Where $|a|$ and $\angle a$ indicate the magnitude and phase angle of the complex number a, respectively. It is necessary to compute $|T|$ and $\angle T$ in terms of $|L|$ and $\angle L$ in order to evaluate these functions

$$L = \Gamma e^{j\phi}$$

Therefore
$$|L| = \Gamma \quad \text{and} \quad \angle L = \phi \tag{3-30}$$

and $$T = \frac{1}{H} \frac{\Gamma e^{j\phi}}{1 + \Gamma e^{j\phi}} \tag{3-31}$$

let $$H = |H| e^{j\underline{/H}} \quad \text{and} \quad G = |G| e^{j\underline{/G}} \tag{3-32}$$

therefore

$$T = \frac{\Gamma}{|H| \sqrt{1 + 2\Gamma \cos\phi + \Gamma^2}} e^{j\left[\phi - \tan^{-1} \frac{\Gamma \sin\phi}{1 + \Gamma\cos\phi} - \underline{/H}\right]} \tag{3-33}$$

We are now in a position to compute the functions defined by Equations (3-26) to (3-29). Note that

$$\Gamma = |H| \bullet |G| \qquad d\Gamma = |H| \cdot d|G| \tag{3-34}$$

$$\phi = \underline{/H} + \underline{/G} \qquad d\phi = d\,\underline{/G} \tag{3-35}$$

From Equation (3-26) we have

$$S_M^M = \frac{d|T|/|T|}{d|L|/|L|}$$

$$= |H| \sqrt{1 + 2\Gamma\cos\phi + \Gamma^2} \; \frac{d\left[\dfrac{\Gamma}{|H|\sqrt{1 + 2\Gamma\cos\phi + \Gamma^2}}\right]}{d\Gamma} \tag{3-36}$$

But $|H|$ is not a function of the variable part of Γ then

$$S_M^M = \sqrt{1 + 2\Gamma\cos\phi + \Gamma^2} \; \frac{d\left[\dfrac{\Gamma}{\sqrt{1 + 2\Gamma\cos\phi + \Gamma^2}}\right]}{d\Gamma}$$

$$= \frac{1 + \Gamma\cos\phi}{1 + 2\Gamma\cos\phi + \Gamma^2} \tag{3-37}$$

By similar technique we find

$$S_A^A = \frac{1 + \Gamma\cos\phi}{1 + 2\Gamma\cos\phi + \Gamma^2} \tag{3-38}$$

$$S_A^M = \frac{\Gamma\sin\phi}{1 + 2\Gamma\cos\phi + \Gamma^2} \tag{3-39}$$

$$S_M^A = \frac{-\Gamma\sin\phi}{1 + 2\Gamma\cos\phi + \Gamma^2} \tag{3-40}$$

From above results it is shown that:

$$\mathrm{Re}\left[S_o^c\right] = S_M^M = S_A^A \tag{3-41}$$

$$\mathrm{Im}\left[S_o^c\right] = S_M^A = -S_A^M \tag{3-42}$$

S_M^M, S_A^M, S_M^A, and S_A^A are all real numerics which are desired small, ideally zero. To investigate the utility of the sensitivity function, the mapping of the loci of constant S_M^M and S_M^A on the polar plane (hence all four) will be considered.

B - Mapping of S_M^M Loci onto the Polar Plane.

We require

$$\frac{1 + \Gamma \cos \phi}{1 + 2\Gamma \cos \phi + \Gamma^2} = M \tag{3-43}$$

Where M is a constant. Changing to cartesian coordinates we get

$$\frac{1 + x}{1 + 2x + x^2 + y^2} = M \tag{3-44}$$

or

$$\left[x + (1 - \frac{1}{2M})\right]^2 + y^2 = \left(\frac{1}{2M}\right)^2 \tag{3-45}$$

Thus the loci of constant S_M^M are a family of circles with center coordinates

$$x = \frac{1}{2M} - 1 \quad \text{and} \quad y = 0 \tag{3-46}$$

and

$$\text{radius} = \frac{1}{2M} \tag{3-47}$$

C - Mapping of S_M^A Loci onto the Polar Plane.

We require

$$\frac{-\Gamma \sin \phi}{1 + 2\Gamma \cos \phi + \Gamma^2} = Q \tag{3-48}$$

where Q is a constant. Changing to cartesian coordinates we get

$$Q = \frac{-y}{1 + 2x + x^2 + y^2} \tag{3-49}$$

or

$$(x + 1)^2 + (y + \frac{1}{2Q})^2 = \left(\frac{1}{2Q}\right)^2 \tag{3-50}$$

Thus the loci of constant S_M^A are a family of circles with center coordinates

$$x = -1, \quad y = -\frac{1}{2Q} \tag{3-51}$$

and

$$\text{radius} = \frac{1}{2Q} \tag{3-52}$$

Both families of circles derived for S_M^M and S_M^A are plotted in Figure 3-7 and it is observed that the region of poorest sensitivity is near the (-1, 0) point, the sensitivity improves in every radial direction from that point.

D - Mapping of $\left| S_o^C \right|$ Loci onto the Polar Plane.

From Equation (3-25) we require

$$\frac{1}{\sqrt{1 + 2\Gamma\cos\phi + \Gamma^2}} = M \tag{3-53}$$

where M is a constant. In cartesian coordinate it is

$$(x+1)^2 + y^2 = \left(\frac{1}{M}\right)^2 \tag{3-54}$$

The loci of constant $\left| S_o^C \right|$ are a family of circles with center coordinates,

$$x = -1, \quad y = 0 \tag{3-55}$$

and

$$\text{radius} = \frac{1}{M}$$

E - Mapping of $\angle S_o^C$ Loci onto the Polar Plane.

From Equation (3-25) we require that

$$\tan^{-1} \frac{\Gamma \sin\phi}{1 + \Gamma\cos\phi} = C' \tag{3-57}$$

or in cartesian coordinate

$$y = \tan C' \,(1 + x) = C(1 + x) \tag{3-58}$$

The loci of constant phase angle of the sensitivity function are therefore straight lines through the point (-1, 0) with slope C. This family of lines comprises the orthogonal trajectories to the loci of constant $\left| S_o^C \right|$.

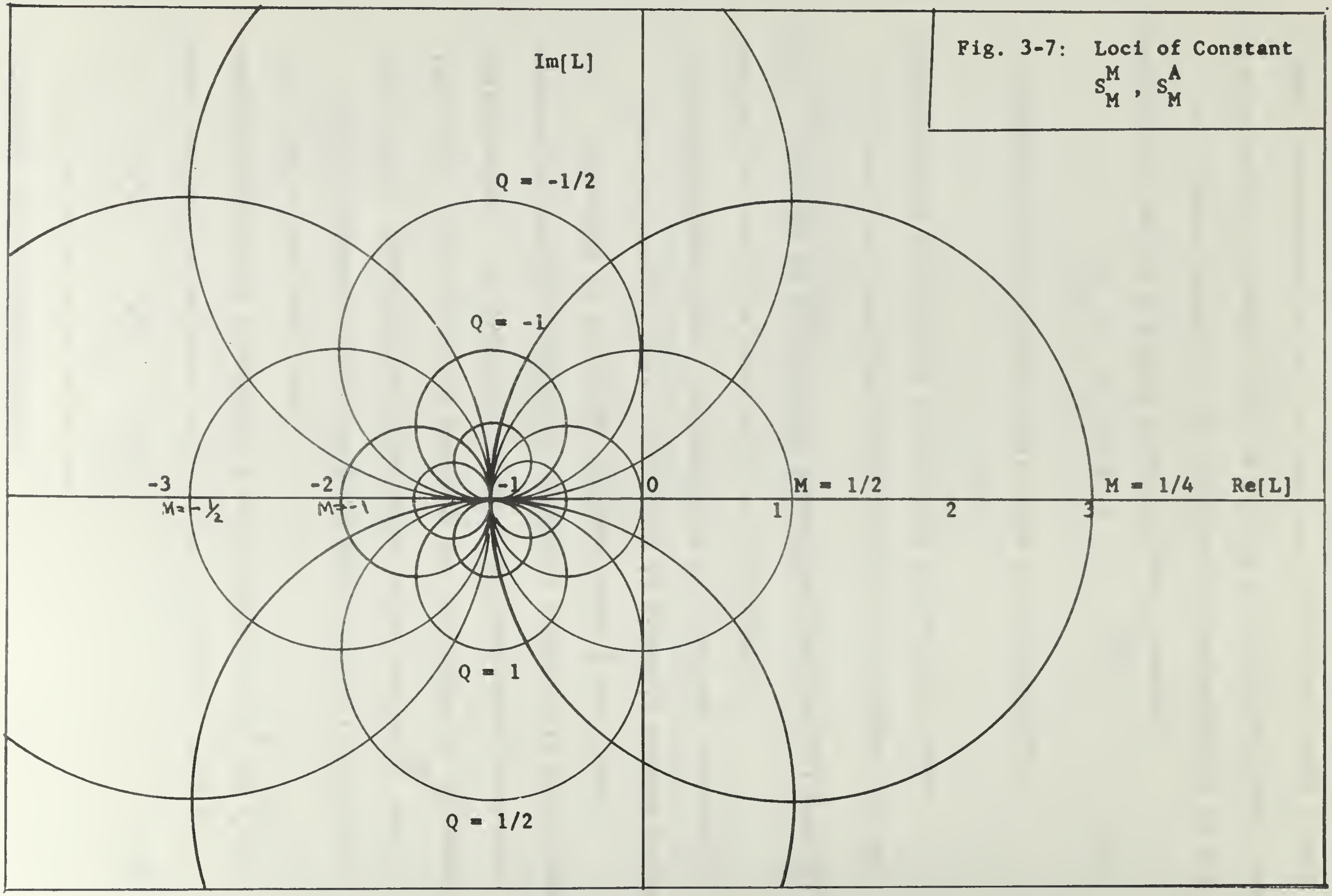


Fig. 3-7: Loci of Constant S_M^M, S_M^A

F - Result of the Interpretation.

For any point on the polar plane

$$\left(S_M^M\right)^2 + \left(S_M^A\right)^2 = \left|S_0^C\right|^2 \tag{3-59}$$

where each term is evaluated at the point in question. Thus the magnitude sensitivity function is a vector measure of S_M^M and S_M^A at every point of the polar plane.

The only extent to which the phase angle loci will be of use comes from the consideration of the phase angles zero and $\frac{\pi}{2}$ rad. Along each of these lines the imaginary and real parts of S_0^C respectively are zero. We conclude that along the phase angle locus corresponding to zero rad., magnitude changes in L cause only phase changes in T. The reverse is true along the phase angle $\frac{\pi}{2}$ rad. locus.

Since frequency response design work is generally carried out on the amplitude-phase plane, the mapping of the magnitude sensitivity function loci onto this space will now be considered.

III-4 - Mapping of the Magnitude Sensitivity Function Loci onto the Amplitude-Phase Plane.

The magnitude sensitivity function loci can be mapped onto the amplitude-phase plane simply by inserting the variable change

$$Z = \ln \Gamma$$

into Equation (3-53). There results from this substitution,

$$e^{2Z} + 2e^{Z} \cos \phi + 1 = \left(\frac{1}{M}\right)^2 \tag{3-61}$$

Plotting this on the Z-ϕ plane gives the desired mapping.

The ultimate purpose for the creation of this mapping is to use it as an overlay in design, just as is used in the Nichols Chart. By a simple manipulation it will be shown that the Nichols Chart itself is already calibrated for our use. In previous sections we concluded

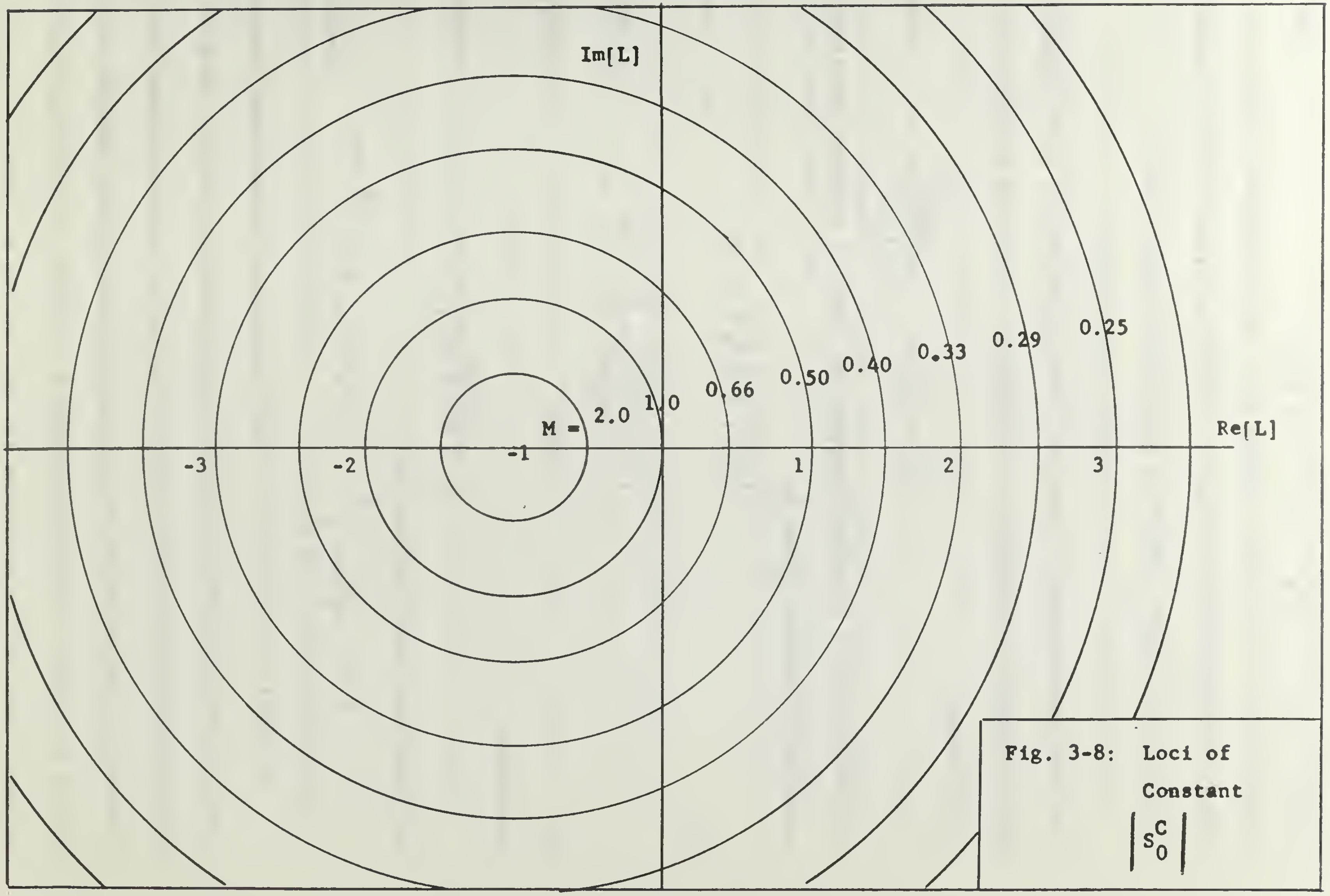


Fig. 3-8: Loci of Constant $\left|S_0^C\right|$

$$S_K^T = S_0^C = \frac{1}{1 + L} \tag{3-62}$$

where, $L(s) = G(s)\,H(s)$, Figure 3-5. It is intended to point out that this sensitivity function is obtainable directly from a conventional Nichols Chart, given the open-loop frequency locus

$$L(j\omega) = G(j\omega)\,H(j\omega)$$

for the linear system in question. The procedures is quite direct. Both numerator and denominator of Equation (3-62) may be divided by $L(s)$ giving,

$$S_0^C = S_K^T = \frac{L(s)^{-1}}{1 + L(s)^{-1}} \tag{3-63}$$

We seek the polar coordinate mapping of lines of constant magnitude sensitivity function given by

$$\left| S_0^C \right| = \left| S_K^T \right| = M$$

or

$$\left| \frac{L(s)^{-1}}{1 + L(s)^{-1}} \right| = M \tag{3-64}$$

let

$$L(s)^{-1} = s + jy \tag{3-65}$$

therefore

$$\left| \frac{x + jy}{1 + x + jy} \right| = M \tag{3-66}$$

and from Equation (3-66) we get

$$\left(x + \frac{M^2}{M^2 - 1} \right)^2 + y^2 = \left(\frac{M}{M^2 - 1} \right)^2 \tag{3-67}$$

We recall the equation for the M circles of conventional design (ref. 9) and note that it is identical to Equation (3-67). Thus we have the interesting result that the M-circles are to the open-loop frequency response curves as the magnitude sensitivity function loci are to the inverse open-loop frequency response. The Nichols Chart is just a plot of the M-circles (as well as the N-circles of constant

closed-loop phase) on the amplitude-phase plane, therefore the mapping of the magnitude sensitivity function loci onto the maplitude-phase plane is the Nichols Chart, provided that we read intersections as derived from inverse open-loop frequency response curves.

The combination of the logarithmic amplitude scale and linear phase scale on the amplitude phase plane, with the symmetric Nichols chart is shown on Figure 3-9. Consider a typical open-loop locus and its inverse, on the extended Nichols chart as in Figure 3-9. The inverse function $L^{-1}(j\omega)$ is obtained on a point by point basis graphically and is symmetric with the direct function $L(j\omega)$ with respect to the point (0 DB, o deg). The Nichols chart is shown over a 540 degree range and is seen to be symmetric (Ref. 9) about the axis phase-0 deg.

One further simplification may be made as shown in Figure 3-10 If we imagine that the phase calibration for $L^{-1}(j\omega)$ locus on the Nichols Chart is opposite to that commonly used then the same M-contrours used in ordinary control system design, measure sensitivity function magnitude for use to the extent of being properly labelled. In this case $L(j\omega)$ and $L^{-1}(j\omega)$ are symmetric with respect to zero DB line, with phase measured in opposite directions for each. For the particular example of Figure 3-10 we see that the maximum magnitude sensitivity function value is $M = 2.0$.

If it was desired, the magnitude sensitivity function for the closed loop system as a function of frequency could be read directly off the Nichols Chart as the intersections of the M-contours with the inverse open-loop frequency response curve.

The simplification introduced by foregoing interpretation of the sensitivity function magnitude and phase loci could be very useful in the design of linear systems with regards to sensitivity specifications.

Special Case:

If the forward elements can be separated into two tandem parts, one of which displays sensitivity to the environment (denoted by an "environmental parameter" x), viz.,

$$G(s, x) = G_1(s)\, G_2(s, x) \tag{3-68}$$

Then we may compute the sensitivity of the closed loop locus of the overall system to the environmental parameter (denoted S_x^T) according to

$$\begin{aligned} S_0^C &= \frac{dT/T}{dL/L} = \frac{dT/T}{dx/x}\,\frac{dx/x}{dL/L} \\ &= S_x^T\, S_0^x \end{aligned} \tag{3-69}$$

or

$$S_x^T = S_0^C \frac{1}{S_o^x} = S_0^C\, S_x^0 \tag{3-70}$$

In the present example

$$\begin{aligned} S_x^o &= \frac{dL/L}{dx/x} = \frac{dG_2(s,x)/G_2(s,x)}{dx/x} \\ &= S_x^{G_2} \end{aligned} \tag{3-71}$$

Therefore

$$S_x^T = S_0^C\, S_x^{G_2} \tag{3-72}$$

where the first term of the product is evaluated graphically, and the second term is found by differentiation. It should be noted that if the parameter x appears in $G_2(s)$ as $G_2(s) = xG_3(s)$, then $S_x^{G_2} = 1$ and $S_x^T = S_0^C$, which is in accordance with our relation $S_0^C = S_k^T$, where k appears as in Figure A1-1 of Appendix 1.

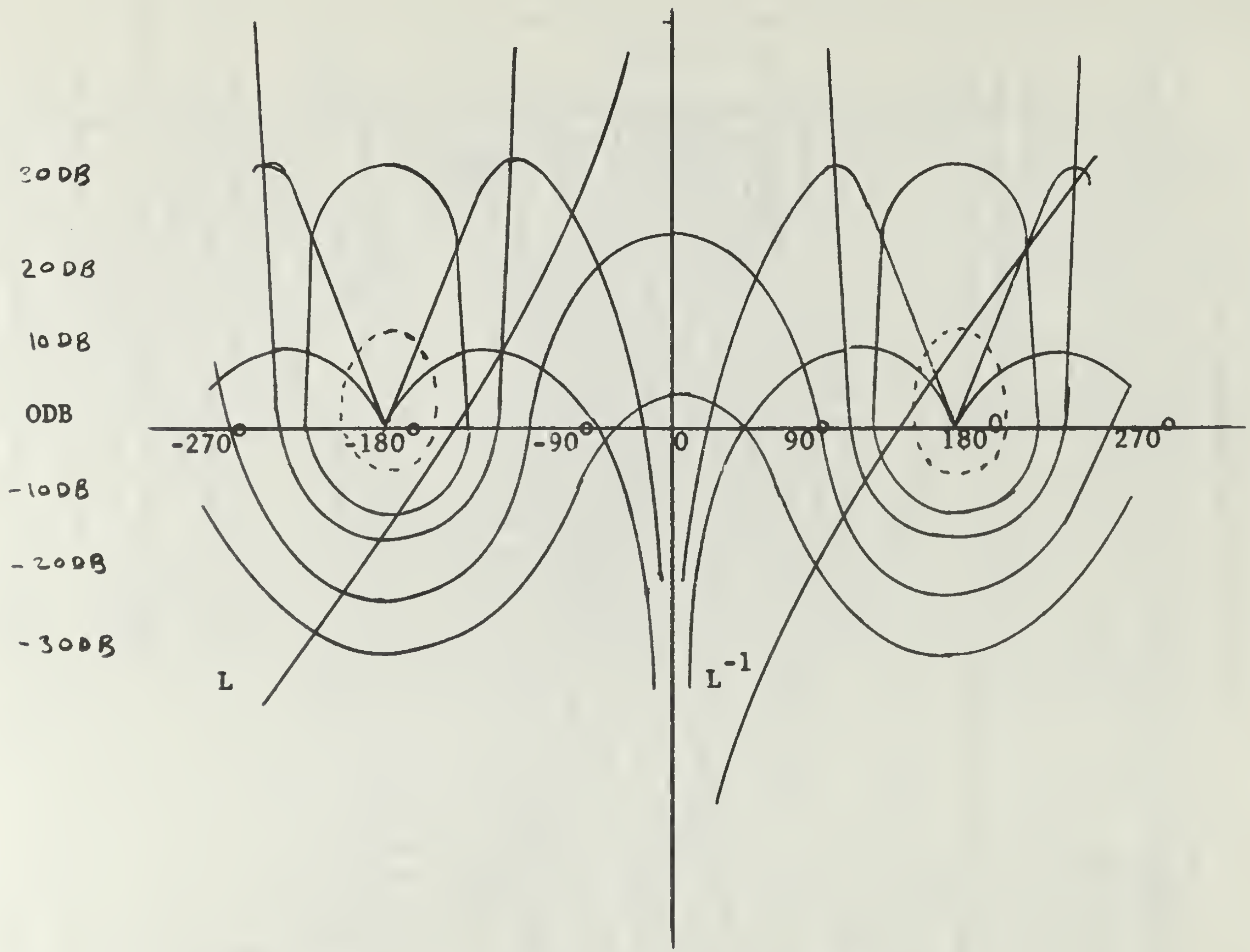


Fig. 3-9: Typical L and its Inverse on Extended Nichols Chart.

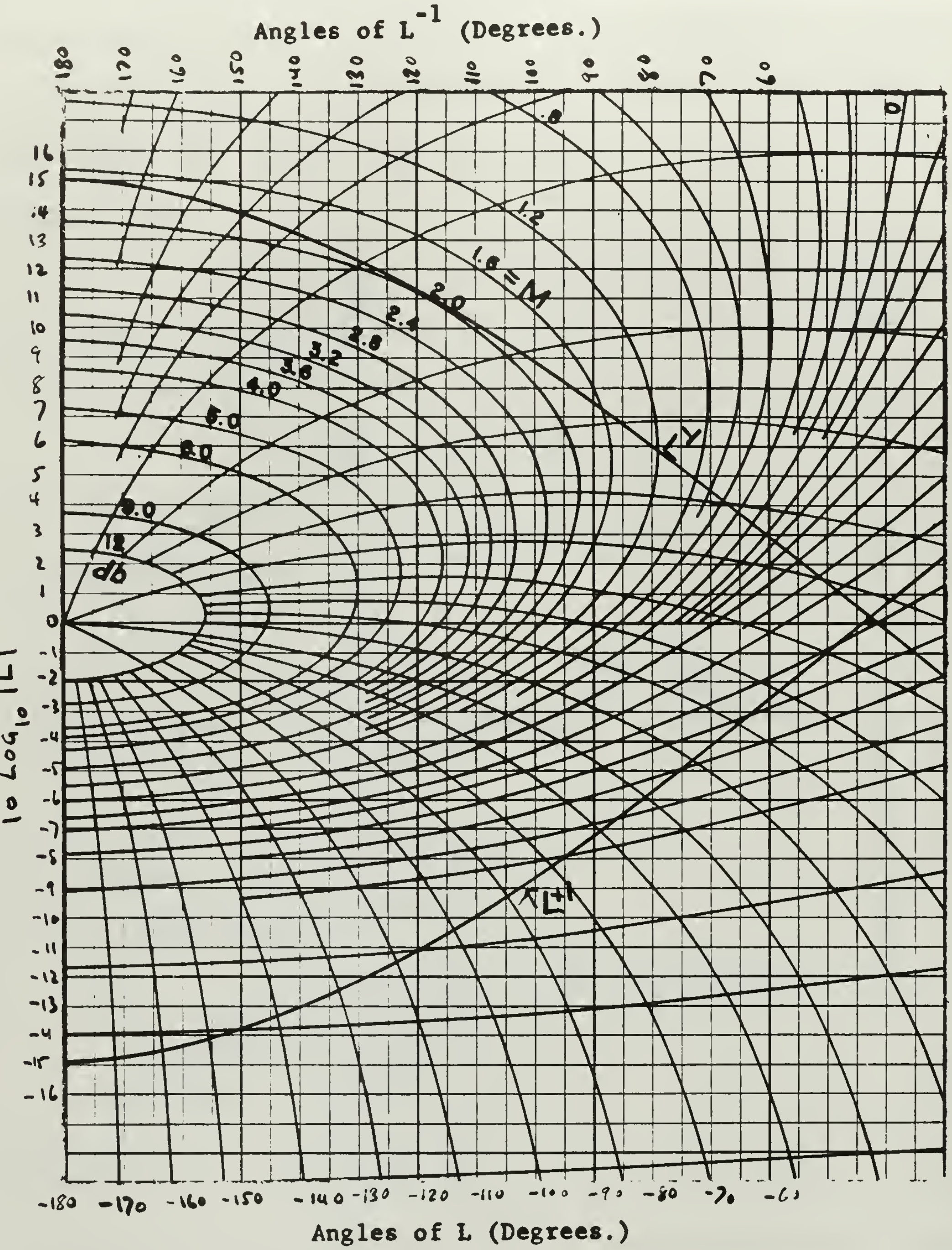


Fig. 3-10: L and L^{-1} on the Nichols Chart.

III-5 - Philosophy of the Frequency Response Approach to Sensitivity Problem.

Case 1 - When $t_{oi} = 0$.

In previous sections we discussed how sensitivity and frequency are related and how by utilizing the Nichols Chart, the magnitude and phase of the sensitivity function and its peak magnitude can be obtained. Another approach which could be of high importance in most design cases is discussed as follows:

Sensitivity specifications could be given in terms of the systems behavior on the $j\omega$ axis. It is possible that the allowed variation in $T(j\omega)$ may be exceedingly small over one frequency range and fairly large in other frequency ranges. The specification may obviously be in a wide variety of forms. For example the nominal T_0, may be as shown in Figure 3-11, with bandwith ω_1, amplitude peaking M_o, and the specification may dictate that despite parameter variations, the magnitude peaking should never exceed 30 percent, the bandwidth should be in the range $0.8\omega_1$ to $1.2\omega_1$, and that from zero frequency to $0.5\omega_1$, $|T|$ should never be less than 0.95. Finally, some thought should be given to the intermediate and the far off frequency ranges. For example, the behavior shown by dotted lines in Figure 3-11 is usually intolerable, as it results in high frequency ringing. A reasonable statement might be that for $1.2\omega_1 < \omega < \omega_3$, $|T(j\omega)| < 3DB$ and for $\omega > \omega_3$, $T(j\omega)$ must not exceed a predetermined value.

The basic philosophy of the frequency response approach to sensitivity reduction is deduced from Equation 3-17, rewritten here

$$\frac{T_o}{T_f} = \frac{(k_o/k_f) + L_o}{1 + L_o} \qquad (3\text{-}73)$$

This equation applies whenever the leakage transmission t_{oi} is zero

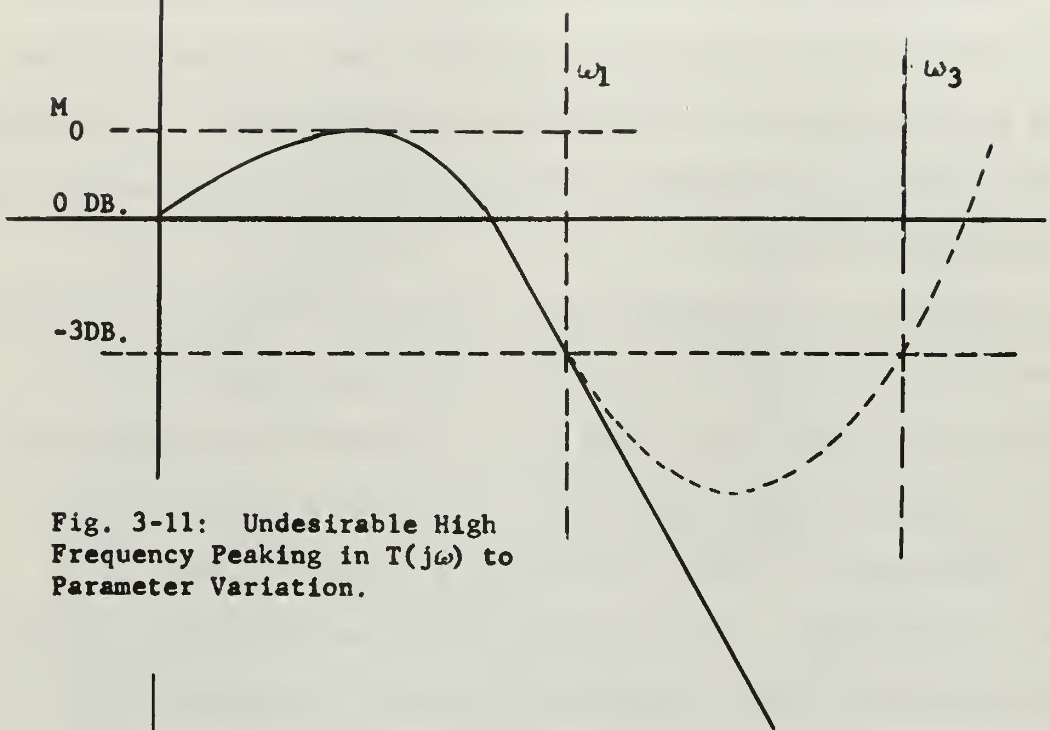


Fig. 3-11: Undesirable High Frequency Peaking in $T(j\omega)$ to Parameter Variation.

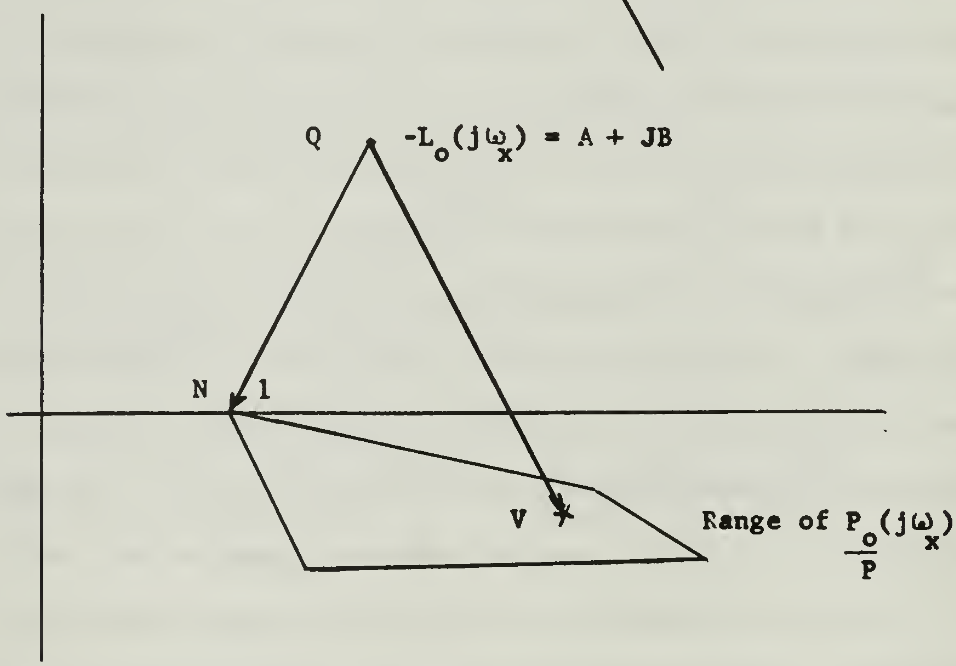
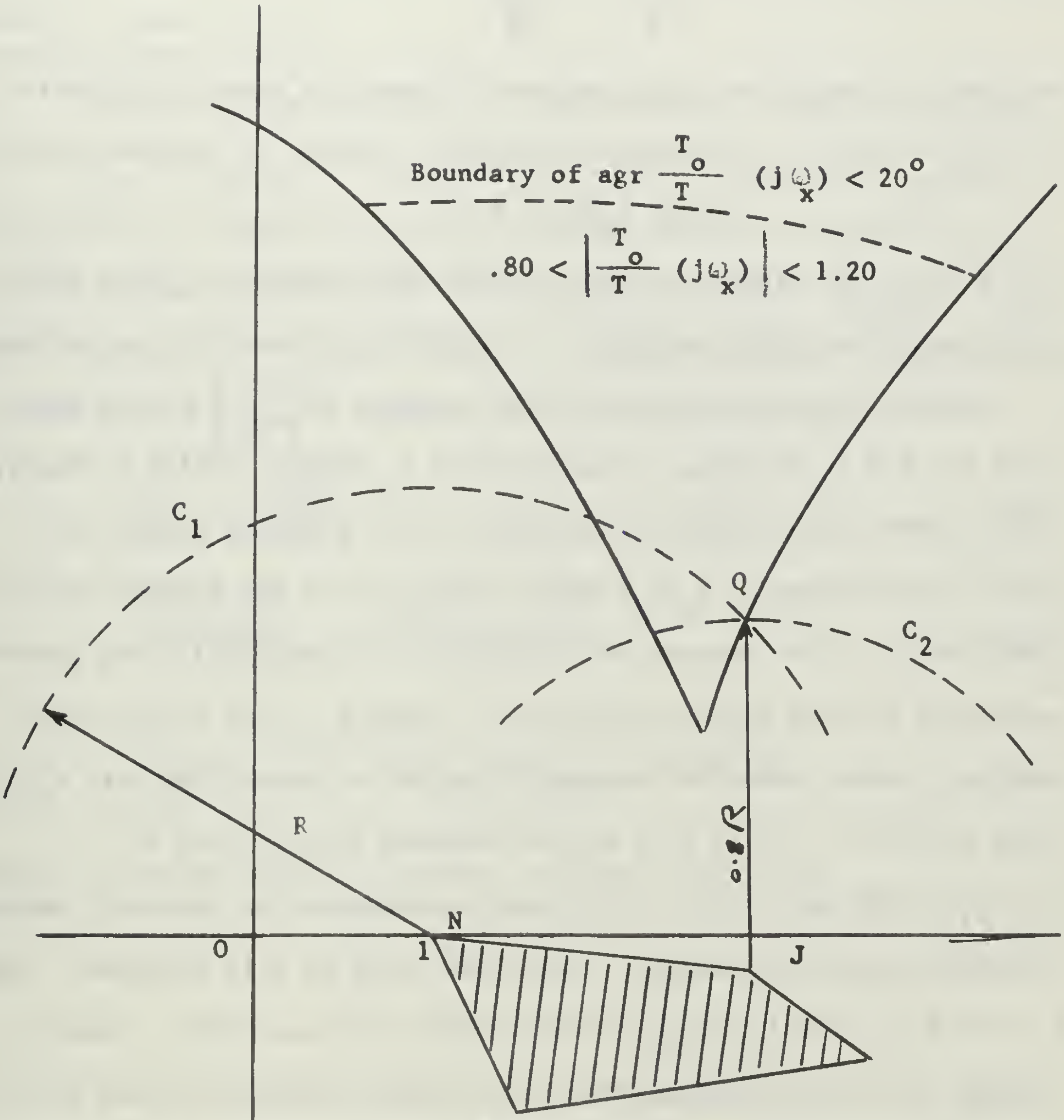


Fig. 3-12: Showing $\frac{T_o}{T_f} = \frac{QV}{QN}$

Fig. 3-13: To Obtain Permissible Location of $-L_o(j\omega_x)$



Range of $P_o/P(j\omega_x)$.

and therefore could be used in most control problems. Suppose that at some specific frequency $s = j\omega_x$, plant parameter variations are such that (k_o/k_f) at $s = j\omega_x$ may lie anywhere inside the region indicated in Figure 3-12. Suppose also that $-L_o(j\omega_x)$ is given by the complex number $A + jB$ located at Q. Then Equation (3-73) states that

$$\frac{T_o}{T_f} = \frac{\overrightarrow{QV}}{\overrightarrow{QN}}$$

The range of variation of the vector QV fixes the range of variation of T_o/T_f at $s = j\omega_x$. Suppose for example, that it is required that:

$$1.20 > T_o/T_f > 0.80$$

at $s = j\omega_x$. The problem is then to find the locus of $-L_o(j\omega_x)$ which barely satisfies this inequality. A suitable procedure is as follows:

Suppose we seek the locus of the boundary of $\left|\frac{T_o}{T_f}\right| \geq 0.80$ with point $N = 1 + j_o$ as center, draw a circle C_1 of any radius R (Figure 3-13). Next draw a circle C_2 of radius 0.8R, using as center the point (the boundary of k_o/k_f) which appears to be the closest to the first circle - this appears to be point J in Figure 3-13. The intersection of the two circles determines a point Q. As a final check, use Q as center, and check whether a circle of radius 0.8R cuts k_o/k_f locus only at J. If so Q is on the boundary of the locus of $\left|T_o/T_f\right| \geq 0.80$ at $s = j\omega_x$. The above construction is repeated, using different values for radius R, until the locus of Q is obtained. Such a locus for $1.20 > \left|T_o/T_f\right| > 0.80$ is shown in Figure 3-13. Clearly, $-L_o(j\omega_x)$ must be located inside the indicated region in Figure 3-13. On the other hand if there is the additional requirement that the phase of T_o/T_f must not change more than 20^o, then the permissible range of $-L_o(j\omega_x)$ is reduced to the smaller region shown in Figure 3-13.

It is extremely important to note that due to Equation (3-17) the sensitivity requirements essentially determine the gain and bandwidth of $L_o(s)$ with hardly any need to refer to the specific feedback configuration that is used. At best one configuration may be more efficient than another in the practical realization of the required $L_o(s)$. The latter is actually a problem in network synthesis.

Case 2 - when $t_{oi} \neq 0$.

In case 1 we assumed that in the fundamental feedback equation $T = t_{oi} + kt_{ci}t_{os}/(1 - kt_{cs})$, the leakage transmission t_{oi} was negligible in comparison with T, over the range of variation of k. As previously discussed this is not always the case. When the leakage transmission t_{oi} was not small in comparison with T, then Equations (3-18) and (3-20) were resulted which are rewritten here:

$$S_k^T = \frac{\Delta T/T_f}{\Delta k/k_f} = \frac{1}{1 + L_o}\left(1 - \frac{t_{oi}}{T_f}\right) \tag{3-74}$$

$$\frac{T_o - t_{oi}}{T_f - t_{oi}} = \frac{(k_o/k_f) + L_o}{1 + L_o} \tag{3-75}$$

$$\frac{T_f}{T_o} = \frac{(1 + L_0) - (t_{oi}/T_o)(1 - k_o/k_f)}{L_o + k_o/k_f} \tag{3-76}$$

where T_f is the value of T when k has the value k_f and

$$\Delta T = T_f - T_o$$

$$\Delta k = k_f - k_o$$

Now t_{oi} is a transfer function which is independent of k. If t_{oi} is known and the desired T is known, then one may relate the maximum permissible range of variation of T, over any frequency range, to a corresponding maximum range of variation of $T - t_{oi}$. In other words

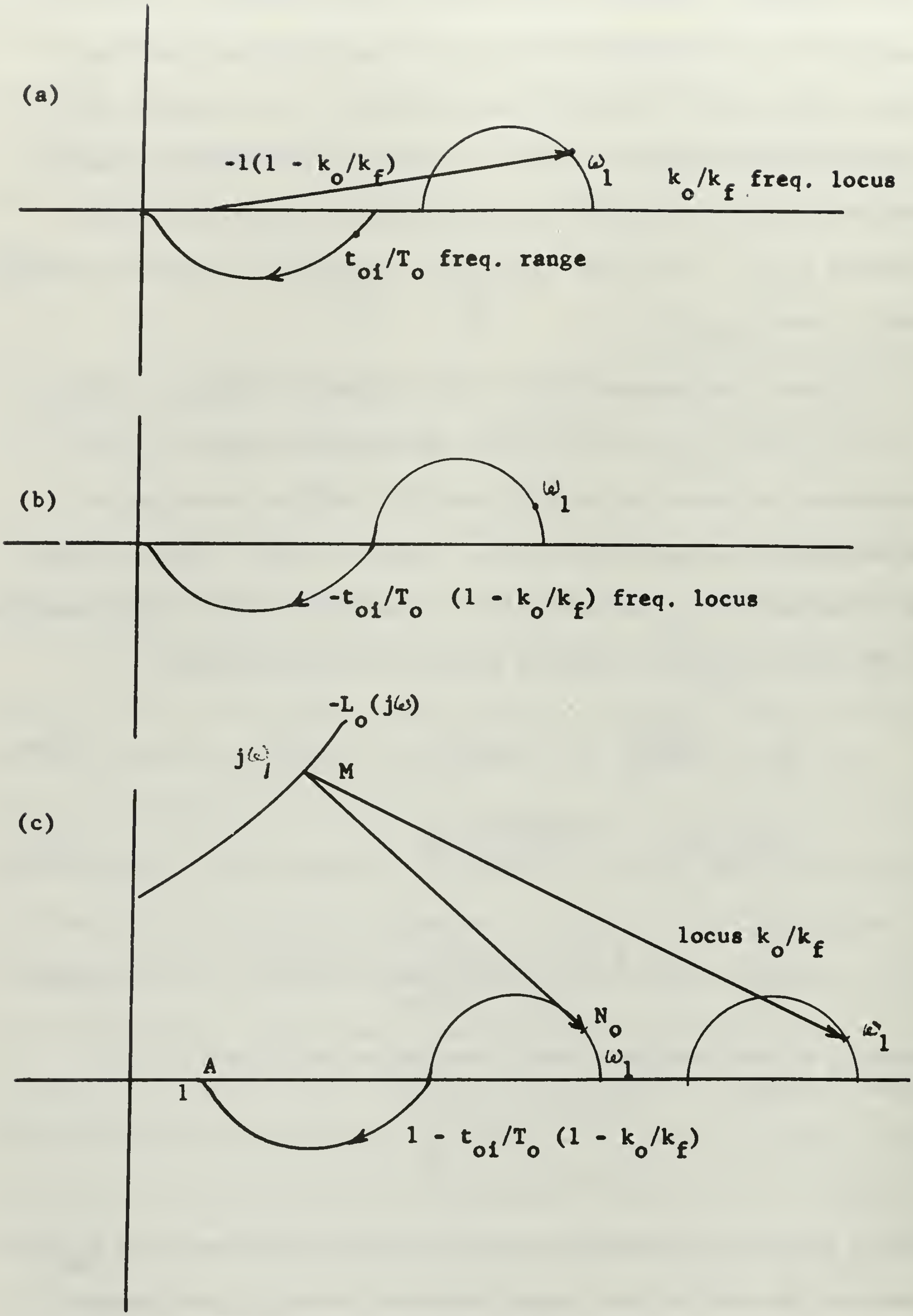


Fig. 3-14: Construction for finding T_o/T_f, when t_{oi} is not zero.

one can work graphically with Equation (3-75) in exactly the same manner as Equation (3-73) was used in Case 1. The only difference is that one must first decide what is the permitted range of variations of $T_f - t_{oi}$, rather than that of T_f.

The above is satisfactory if t_{oi} is known. On the other hand, there may be situations when t_{oi} is not precisely known until part of the design has been done. In such a case some cut and try is inevitable. The following modification of the graphical procedure, suggested by Horowitz is an alternative method of handling the non-zero leakage transmission problem. Equations (3-75) to (3-77) are used. Sketch the frequency locus of t_{oi}/T_o (Figure 3-14a) in the complex plane, over the frequency range in which control of variation in T is desired. Also sketch the loci of k_o/k_f. From these two figures, obtain the loci of $-(1 - k_o/k_f)\ (t_{oi}/T_o)$ as in Figure 3-14b. Finally combine the latter and the sketch of k_o/k_f into one figure (Figure 3-14). In Figure 3-14c we have

$$\frac{T_f}{T_o} = \frac{MN}{MB}$$

in accordance with Equation (3-76). The design problem is to select the loop transmission L_o so as to satisfy the specification of T_o/T_f.

III-6 - Use of Polar Plot in Studying the Sensitivity.

The relation, $$S_K^T = S_0^C = \frac{1}{1 + L_o(s)}$$

suggests that the inverse polar plot could be useful in studying the sensitivity function. Above relation could be written as

$$S_K^T = S_0^C = \frac{1}{1 + L_o} = \frac{L_o^{-1}(s)}{1 + L_o^{-1}(s)} \qquad (3-78)$$

Referring to Figure 3-15 it is noticed that if OP represents $L_o^{-1}(j\omega)$, then $BP = 1 + L_o^{-1}(j\omega)$ and,

$$S_O^C = \frac{L_o^{-1}(j\omega)}{1 + L_o^{-1}(j\omega)} = \frac{OP}{BP} = Me^{j\alpha} \tag{3-79}$$

As discussed in Equations (3-63) to (3-67) we have

$$\left[x - \frac{M^2}{1 - M^2}\right]^2 + y^2 = \left(\frac{M}{1 - M^2}\right)^2 \tag{3-80}$$

where $\quad x$ = real part of $L_o^{-1}(j\omega)$,

$\quad y$ = imaginary part of $L_o^{-1}(j\omega)$.

Equation (3-80) represents a circle having radius $= \dfrac{M}{1 - M^2}$, and

center at $\dfrac{-M^2}{M^2 - 1}$ on the x axis. (3-81)

The loci of constant M described above is useful for determining the value of M_{max} for a control system for which $L_o(s)$ and eventually $L_o^{-1}(s)$ is known. The radius of the M circle and the distance from the origin to the center of a given M circle are both a function of M alone. Referring to Figure 3-16, where P is the point of tangency from the origin to the M circle, one observes that the ratio of BP to OB is a constant for a given M.

$$BP = \text{radius of M circle} = M/(M^2 - 1),$$

$$OB = M^2/(1 - M^2),$$

so

$$\sin\Psi = \frac{\dfrac{M}{1 - M^2}}{\dfrac{M^2}{1 - M^2}} = \frac{1}{M}$$

Then $\sin\Psi$ is solely defined in terms of M.

$$\frac{OC}{OB} = \frac{OP\cos\Psi}{OB} = \frac{OB\cos^2\Psi}{OB} = 1 - \frac{1}{M^2} = \frac{M^2 - 1}{M^2}$$

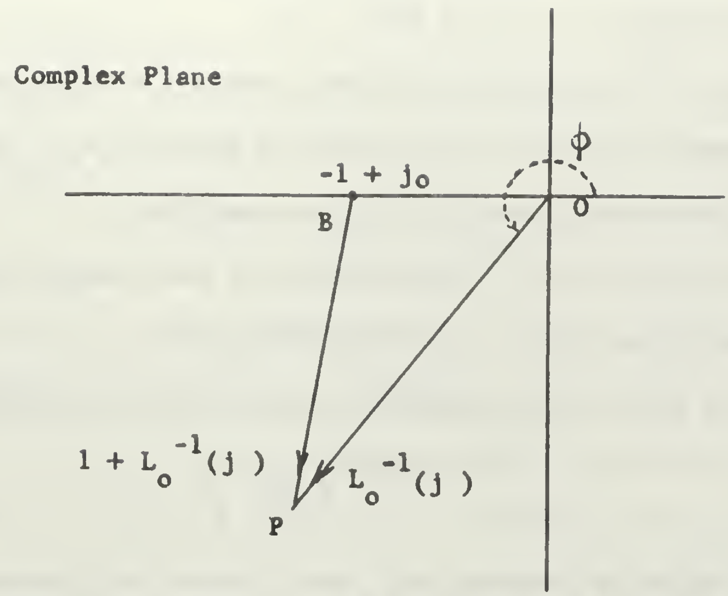


Fig. 3-15: Conventional Form of Complex Plane Diagram for Control System of Equation (3-78).

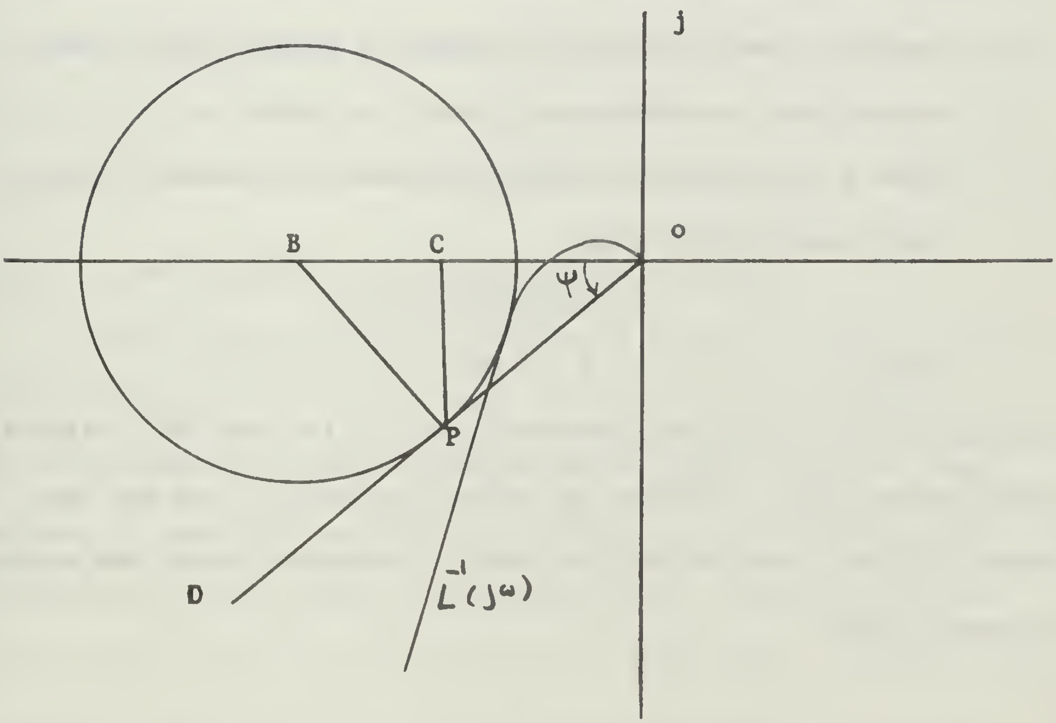


Fig. 3-16: Geometry of M Circles.

or $\quad OB/OC = M^2/(M^2 - 1).$

Hence C represents the location of $-1 + j0$, point to the same scale for which BP represents the length $M/M^2 - 1$.

For a given $L_o^{-1}(s)$ function to have a maximum M value of M_m it must also be tangent to the M circle having a value of M_m. By using these facts the following procedure can be formulated.

1 - Draw the locus of $L_o^{-1}(j\omega)$ function on the complex plane with the value of $K = 1$, K being the gain.

2 - With the value of M_m specified, determine the angle ψ from

$$\psi = \sin^{-1}\left(\frac{1}{M_m}\right)$$

3 - Using the angle ψ determined from 2, draw the line OD on the complex plane.

4 - By cut-and-try, draw the circle that has its center on the negative real axis and is tangent to both $L_o^{-1}(j\omega)$ locus and to the line OD at some point P as shown.

5 - Draw a perpendicular PC from the point of tangency P to the negative real axis.

$$OC(1/K) = 1.0$$

then $$K = OC$$

This value of K is the gain associated with $L_o(s)$ that will yield the specification of M_m. Knowing the proper net gain K, one may then change by K the scale of the plot used in preceding steps and evaluate the sensitivity.

IV

SPECIAL TOPICS

IV-1 - Discussion.

In previous sections we discussed the sensitivity function and showed that:

$$S_k^T = \frac{1}{F_k}\left[1 - \frac{t_{oi}}{T}\right] \tag{4-1}$$

where F_k is the return difference with respect to the parameter k. The parameter k could be any parameter such as gain K, pole P_i, or zero Z_i, and t_{oi} is the leakage or direct transmission between input and output, as shown in Figure 4-2. Although the sensitivity function has been defined and several of its properties examined, it is not yet apparent how one may utilize it in system design, or more fundamentally on what basis its characteristics should be chosen. Similarly in order to compare the relative sensitivies of different parameters it is convenient to have an overall measure of sensitivity which is independent of either time or frequency.

IV-2 - Specification of the Sensitivity Function.

A common design approach consists of making the "loop gain" or return difference, extremely large in the band of frequencies in which the input signals are expected to be. Then the open loop response is caused by appropriate shaping to fall off as rapidly as possible to unity gain outside this band. A maximum roll off rate of about 33 db per decade permits unconditional stability to be achieved, while a greater rate results in conditional stability. Thus realization of low response sensitivity by making the magnitude of the sensitivity function small in the desired band of frequencies

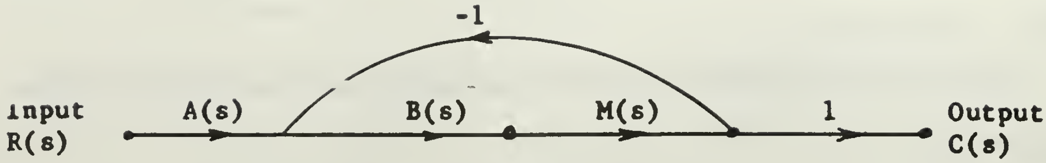


Fig. 4-1: A Basic Feedback System.

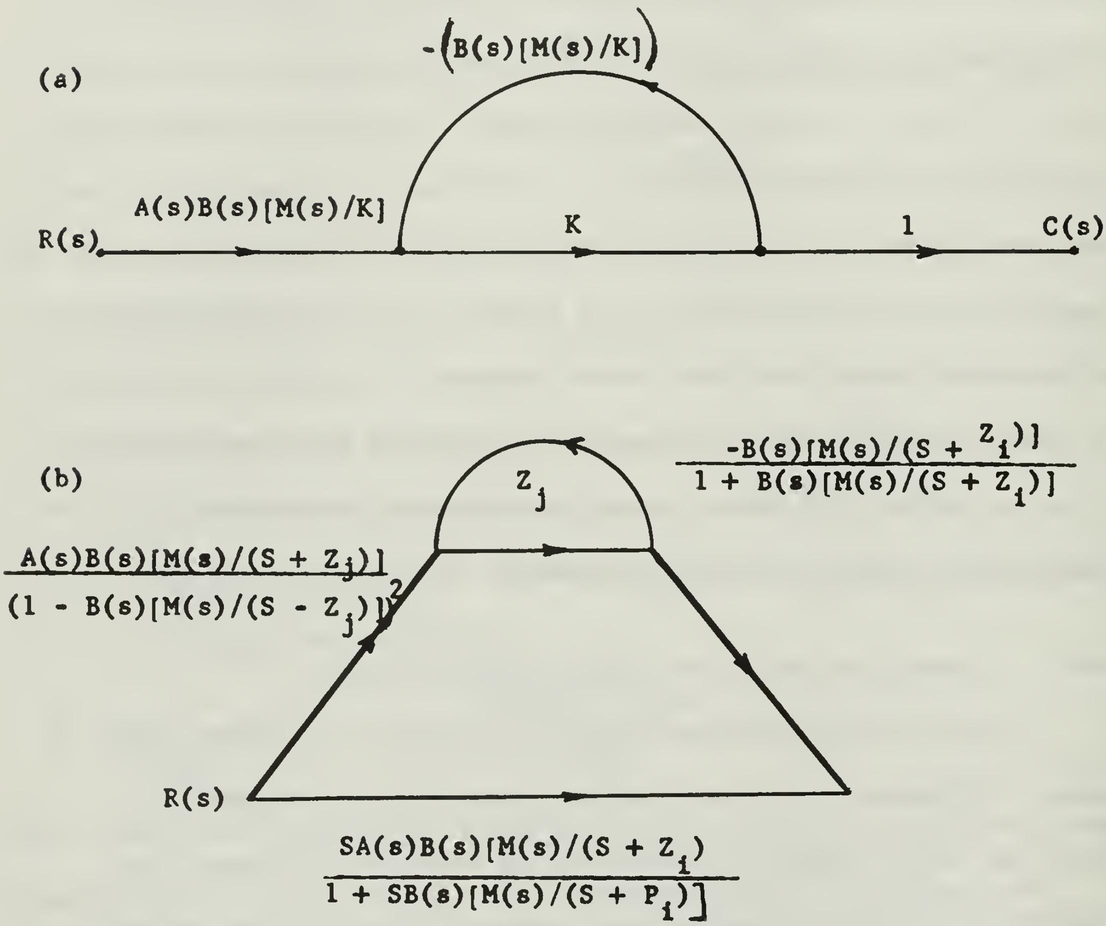
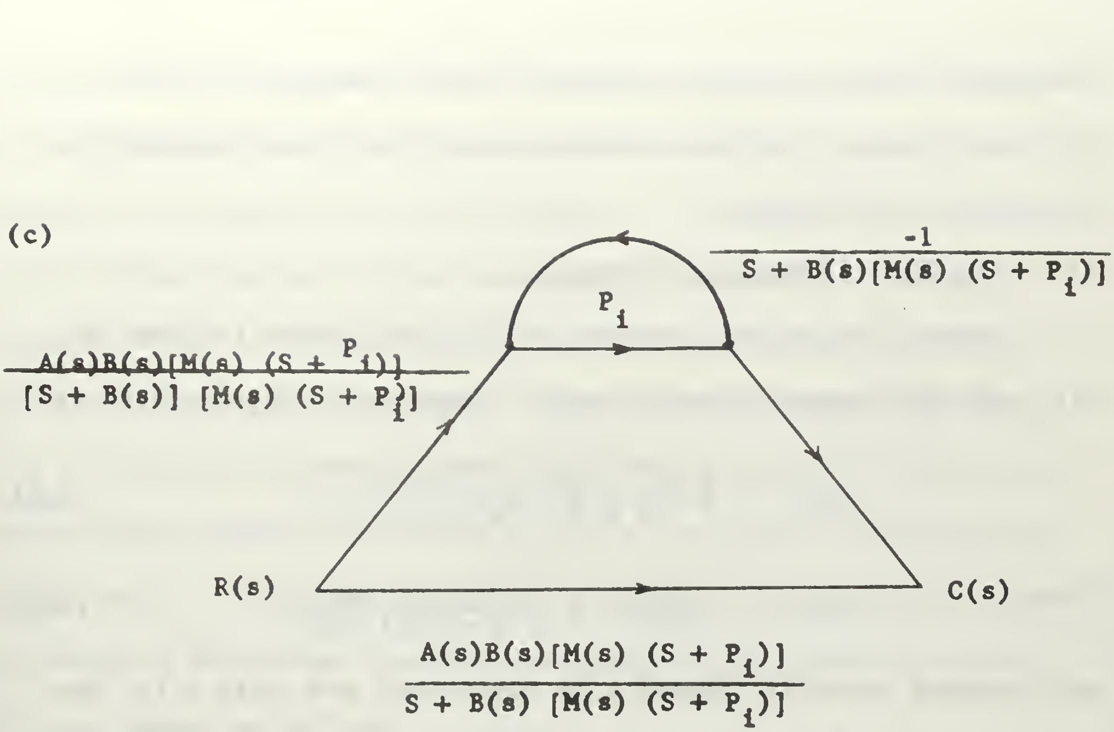


Fig. 4-2: Alternate System Configuration of Figure 4-1.

(c)



(d)

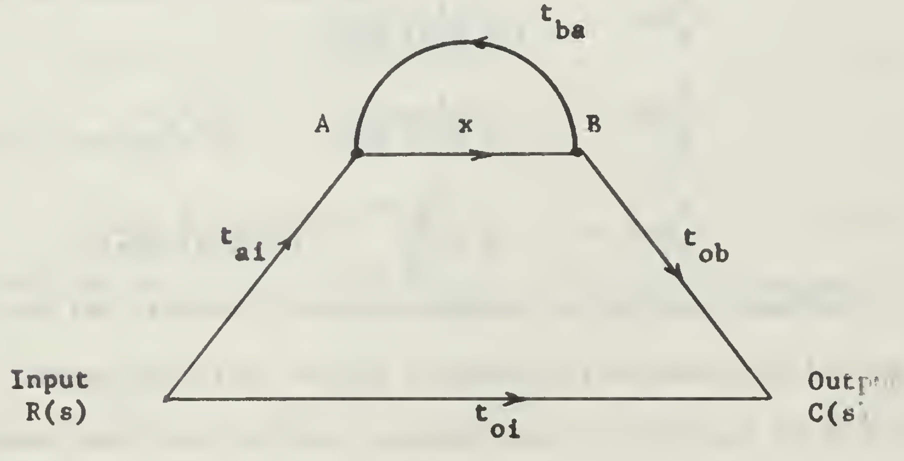


Fig. 4-2: Continued.

inevitably results in the problem of system stability. In the following several criterions for the specification of sensitivity functions are discussed.

(a) - <u>The Forced Response Criterion</u>.

Suppose the desired response in the time domain is given by c(t) and the response error by e(t). Referring to Figure 4-1, if

$$M(s) = K \frac{(s + Z_1)\ (s + Z_2)\ ----}{(s + P_1)\ (s + P_2)\ ----} \tag{4-2}$$

then

$$T(s) = \frac{C(s)}{R(s)} = \frac{A(s)\ B(s)\ M(s)}{1 + B(s)\ M(s)} \tag{4-3}$$

and assuming variable element x is associated with M(s) only, then

$$S_x^T(s) = \frac{x}{T(s)} \frac{dT(s)}{dx} = \frac{x}{M(s)} \frac{dM(s)/dx}{1 + B(s)\ M(s)} \tag{4-4}$$

for x = K, Z_j or P_i Equation (4-4) becomes,

$$S_K^T(s) = \frac{1}{1 + B(s)\ M(s)} \tag{4-5}$$

$$S_{Z_j}^T(s) = \frac{1}{1 + B(s)\ M(s)} \frac{Z_j}{s + Z_j} \tag{4-6}$$

$$S_{P_i}^T(s) = \frac{-P_i}{s + P_i} \frac{1}{1 + B(s)\ M(s)} \tag{4-7}$$

The basic problem is determination of criteria for the specification of the sensitivity function for an arbitrary input. After the form of the desired transmission function has been ascertained, based on some dynamic response criterion, Equations (4-5) to (4-7) reveal that the system parameters which remain to be specified are the zeros of the sensitivity function. The poles of the sensitivity function and the transmission function are identical, except for the added poles Z_j and P_i, provided the product A(s) B(s) M(s) in Equation (4-3) is chosen so that no poles are added to those of the reciprocal

of $F_k(s)$. The forced response criteria determines the zero of S(s) so that e(t) behaves in the desired manner over the interval or intervals of interest in the time domain. A considerable literature exists whereby one may relate time domain responses to the complex frequency domain characteristics of a given network configuration. These results are useful when the inputs are restricted types and when the system poles are simple. Functions of third or higher order with multiple poles and possible poles on the $j\omega$ axis are difficult to deal with. If the sensitivity function is to be specified only on the basis of the forced response to a given input, then the zeros of S(s) are chosen as follows:

If R(s) consists of a number of inputs of the form,

$$R_j(s) = a_j \frac{1}{s + s_j} \tag{4-8}$$

then the system output is given by,

$$C_j(s) = a_j \frac{1}{s + s_j} T(s) \tag{4-9}$$

and the forced response is

$$c_j(t) = a_j T(-s_j) e^{-s_j t} \tag{4-10}$$

Taking the derivative of $c_j(t)$ with respect to a variable parameter x we get,

$$\frac{dc_j(t)}{dx} = a_j e^{-s_j t} \frac{dT(-s_j)}{dx} \tag{4-11}$$

But from Equation (4-4) we have

$$\frac{dT(-s_j)}{dx} = S_x^T (-s_j)/x \tag{4-12}$$

Then substituting Equation (4-12) into (4-11) we get

$$\frac{dc_j(t)}{dx} = \frac{a_j}{x} T(-s_j) S_x^T(-s_j) e^{-s_j t} \tag{4-13}$$

Thus, if the sensitivity function $S_x^T(s)$ has a zero at the pole of the transform of the input function, the system forced response is invariant with changes in x. If s_j's are expected to lie in some known range of values, the sensitivity zeros may be distributed accordingly.

The above result is also valid when s_j is imaginary but in that case both the steady state amplitude and phase responses are of interest. For a sinosoidal input, the forced response and its derivatives with respect to x become

$$c_r(t) = \frac{a_r}{s_r} \left| T(js_r) \right| \sin \left[s_r t + \phi\,(js_r) \right] \qquad (4\text{-}14)$$

where

$$\phi(js_r) = \tan^{-1} \frac{\mathrm{Im}T(js_r)}{\mathrm{Re}T(js_r)}$$

and

$$\frac{dc_r(t)}{dx} = \frac{a_r}{s_r} \left\{ \sin\,(s_r t + \phi) \frac{d\left|T(js_r)\right|}{dx} + \left|T(js_r)\right| \frac{d\phi(js_r)}{dx} \cos\,(s_r t + \phi) \right\} \qquad (4\text{-}15)$$

Writing

$$T(js_r) = \left|T(js_r)\right| e^{j\phi} \qquad (4\text{-}16)$$

Then

$$S_x^T(js_r) = \frac{x}{|T|e^{j\phi}} \frac{d[\,|T|\,e^{j\phi}\,]}{dx}$$

$$= \frac{x}{|T|} \frac{d|T|}{dx} + jx \frac{d\phi}{dx}$$

$$= S_R(js_r) + jS_I(js_r) \qquad (4\text{-}17)$$

The R and I subscripts in the last equation refer to the real and imaginary parts of $S(js_r)$, respectively. Thus comparing Equations (4-15) and (4-17), if zeros of S(s) are assigned at $\pm\, js_r$, then both S_R and S_I are zero and $\frac{dc_r(t)}{dx}$ becomes zero and is invariant with changes in x.

In this section we domonstrated how variations in the forced response may be controlled by suitably specifying the sensitivity function. A design based only upon the forced response is likely to result in an unsuitable selection of the zeros of the sensitivity function, if the system is of high order and the input is of a known invariant form. For example if a fifth order system is to be subjected to a sinosoidal input, then five sensitivity zeros must be assigned. Two zeros are given values corresponding to the poles of the input transform, there is no reasonable basis for selecting the remaining three. Furthermore, if interest is centered upon the part of the system response where the transient has an appreciable value, then even the forced response criteria outlined have little value in determining suitable sensitivity zeros. Then another technique based on "minimum mean square error" was proposed.

(b) - <u>The Minimum Mean Square Error Criterion</u>

The minimization of the mean square difference between a desired and an actual response is fully discussed in literature. This technique is directly applied by Mazer to the sensitivity problems. Its main advantage is that the problem is solved entirely in the complex frequency domain, thus the necessity for obtaining the error in the time responses of high-order feedback systems in terms of the unknown zeros of the sensitivity function is avoided. Before proceeding with details, the following definitions are required:

$$T(s) = \frac{N(s)}{D(s)} \tag{4-18}$$

$$S(s) = \frac{N_o(s)}{D(s)} \tag{4-19}$$

$$\Delta T(s) = \frac{\Delta x}{x} T(s)\, S(s) \tag{4-20}$$

$$E(s) = R(s)\Delta T(s) \tag{4-21}$$

Where R(s) is the laplace transform of the input and Δ notation is used to indicate a small finite, rather than differental, change. The mean square error is given by

$$e^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^2(t)dt \tag{4-22}$$

Here e(t) is the inverse transform of E(s). The square of the error is integrated over a time interval of length 2T, its average is taken and the time interval is then allowed to become infinite. It should be noted here that T(s) is the transfer function of the system, while T is the time interval. As shown in reference (2),

$$e^2 = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Delta T(s)\,\Delta T(-s)\,\phi_r(s)ds \tag{4-23}$$

Where $\phi_r(s)$ is the power density spectrum of the input signal. Substituting Equations (4-19) and (4-20) into (3-23) we get

$$e^2 = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left(\frac{\Delta x}{x}\right)^2 \frac{N_0(s)N_0(-s)}{D(s)D(-s)}\, T(s)T(-s)\phi_r(s)ds \tag{4-24}$$

To minimize the mean square error e^2, one takes its partial derivatives with respect to each unknown coefficient of $N_0(s)$ and sets the results equal to zero. This yields a number of linear algebraic equations equal to the number of unknown coefficients. Furthermore if $N_0(s)$ is of even or odd degree, all coefficients of odd or even powers of s must equal zero. In order to obtain a suitable density spectrum for use in Equation (4-24), aperiodic inputs such as steps and ramps, must be modified by being repeated periodically. Although the system is then designed on the basis of this fictitious input, the periodicity of the repeated input signal determines approximately

the interval of time over which the mean square error, for the case of an aperiodic input is minimized.

Example

Design a feedback system in the configuration of Figure 4-1, with the following closed loop transfer function

$$T(s) = \frac{5}{(s^2 + s + 1)(s + 5)} \tag{4-25}$$

Since the system is of the third order, $N_0(s)$ may immediately be written

$$N_0(s) = s^3 + a_2 s^2 + a_1 s + a_o \tag{4-26}$$

If the variable element is the forward gain k, the mean square error becomes:

$$e^2 = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(\frac{\Delta K}{K}\right)^2 \frac{T(s)T(-s)}{D(s)D(-s)}\left[(a_2 s^2 + a_o)^2 - s^2(s^2 + a_1)^2\right]\left[\phi_r(s)ds\right] \tag{4-27}$$

Since e^2 is the sum of two integrals, one may solve for a_2 and a_o separately. Taking partial derivatives with respect to these quantities,

$$\frac{\partial e^2}{\partial a^2} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(\frac{\Delta K}{K}\right)^2 \frac{T(s)T(-s)}{D(s)D(-s)}\left[2(a_2 s^2 + a_o)\, s^2 \phi_r(s)ds\right] = 0 \tag{4-28}$$

and similarly

$$\frac{\partial e^2}{\partial a_o} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(\frac{\Delta K}{K}\right)^2 \frac{T(s)T(-s)}{D(s)D(-s)}\left[2(a_2 s^2 + a_o)\, \phi_r(s)ds\right] = 0 \tag{4-29}$$

Solving Equation (4-28) and (4-29) for a_2 and a_o homogeneous set of equations results, the solution which is

$$a_2 = a_o = 0$$

To solve for a_1,

$$\frac{\partial \overline{e^2}}{\partial a_1} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(\frac{\Delta K}{K}\right)^2 \frac{T(s)\ T(-s)}{D(s)\ D)-s)}\ [-2s^2(s^2+a_1)]\phi_r(s)ds = 0 \tag{4-30}$$

and from that

$$a_1 = \frac{\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{T(s)T(-s)}{D(s)D(-s)}\quad (s^2)(-s^2)\phi_r(s)ds}{\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{T(s)T(-s)}{D(s)D(-s)}\quad (s)(-s)\phi_r(s)ds} \tag{4-31}$$

Thus a_1 is independent of ΔK, provided the latter is small enough for the first order approximation $\Delta T(s)$ given by Equation (4-20). Furthermore $\phi_r(s)$ being positive, a_1 is always positive and the integrands are written as conjugate products in Equation (4-31) to emphasize this fact. The character of the input remains to be specified so that $\phi_r(s)$ may be evaluated. Suppose the input is a sinusoid of frequency $\beta = 1/2$ rad/sec. and that the mean square error of the system response, over one cycle of the driving function is to be minimized for small changes in k. To generate an appropriate repetitive input signal let the input sinusoid be multiplied by a square wave of frequency $\omega_o = 1/4$ rad/sec. and unity height. The Fourier series and power spectrum of the square wave are

$$r_{sq}(t) = \frac{4}{\pi}\left(\sin\frac{1}{4}t + \frac{1}{3}\sin\frac{3}{4}t + \frac{1}{5}\sin\frac{5}{4}t + \cdots\right)$$

and

$$\begin{aligned}\phi_{sq}(\omega) = \frac{1}{4}\left(\frac{4}{\pi}\right)^2 \Big\{ & [\delta(\omega - \tfrac{1}{4}) + \delta(\omega + \tfrac{1}{4})] \\ & + \tfrac{1}{9}[\delta(\omega - \tfrac{3}{4}) + \delta(\omega + \tfrac{3}{4})] \\ & + \tfrac{1}{25}[\delta(\omega - \tfrac{5}{4}) + \delta(\omega + \tfrac{5}{4})] \\ & + \cdots\cdots \end{aligned} \tag{4-32}$$

Here $\delta(\omega \pm n\omega_o)$ is the Dirac delta function. If this square wave is driven through a fictitious transfer function

$$F(s) = \frac{S}{s^2 + \beta^2} = \frac{S}{s^2 + \left(\frac{1}{2}\right)^2} \tag{4-33}$$

then the power spectrum of the input is given by

$$\phi_r(\omega) = \phi_{sq}(\omega) \; F(j\omega)^2$$

$$= \frac{\omega^2}{[\omega^2 - \left(\frac{1}{2}\right)^2]^2} \; \phi_{sq}(\omega) \tag{4-34}$$

With $\phi_{sq}(\omega)$ given in Equation (4-32), the intergrals in Equation (4-31) are evaluated by summing twice the values of the integrands at $\omega = \frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{4}$, etc., since they are even functions and thus always positive. The summations are continued until the desired accuracy is achieved. The values of the integrands fall rapidly with increasing ω, since the degrees of the denominators are much larger than those of the numerators. Using only the first five terms in the Fourier series of Equation (4-32) and substituting T(s), D(s) and $\phi_r(s)$ we get

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{T(s)T(-s)}{D(s)D(-s)} \; (s)^3(-s)^3\phi_r(s)ds = 0.751(10^{-2}) \tag{4-35}$$

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{T(s)T(-s)}{D(s)D(-s)} \; (s)^2(-s)^2\phi_r(s)ds = 1.238(10^{-2}) \tag{4-36}$$

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{T(s)T(-s)}{D(s)D(-s)} \; (s)(-s)\phi_r(s)ds = 2.511(10^{-2}) \tag{4-37}$$

Then by Equation (4-31) we get

$$a_1 = 0.404$$

and the mean square error is

$$e^2 = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left(\frac{\Delta K}{K}\right)^2 \frac{T(s)T(-s)}{D(s)D(-s)} [(-s^2)(s^2+a_1)^2]\phi_r(s)ds \tag{4-38}$$

Equation (4-38) can be written as

$$e^2 = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left(\frac{\Delta K}{K}\right)^2 \frac{T(s)T(-s)}{D(s)D(-s)} (-s^3)(s^3)\phi_r(s)ds$$

$$+ \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left(\frac{\Delta K}{K}\right)^2 \frac{T(s)T(-s)}{D(s)D(-s)} (-s^2)(s^2)2a_1\phi_r(s)ds$$

$$+ \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left(\frac{\Delta K}{K}\right)^2 \frac{T(s)T(-s)}{D(s)D(-s)} (-s)(s)a_1^2\phi_r(s)ds \tag{4-39}$$

and utilizing relations (4-35) to (4-37) we get

$$e^2 = 0.00149 \left(\frac{\Delta K}{K}\right)^2 \tag{4-40}$$

The sensitvity function is then

$$S_K^T(s) = \frac{N_o(s)}{D(s)} = \frac{s(s^2 + 0.494)}{(s^2 + s + 1)\ (s + 5)} \tag{4-41}$$

The above techniques show how complicated the computation could be and in many cases the use of computer is the quickest way. The purpose of the above discussion is not a complete design technique, but to show how different approaches are made to the problem.

For more insight into this problem reference 2 and 11 are recommended.

IV-3 - Equivalent Input Perturliation Signal

As discussed in Appendix 1, for any linear system, the transfer function T(s,x), relating a response to an input function can be expressed in the following bilinear form in terms of a real parameter

x of the system and the polynomials A(s), B(s), C(s), and D(s)

$$T(s,x) = \frac{P(s,x)}{Q(s,x)} = \frac{A(s) + xB(s)}{C(s) + xD(s)} \tag{4-42}$$

where

$$P(s,x) = A(s) + xB(s) \tag{4-43}$$

$$Q(s,x) = C(s) + xD(s) \tag{4-44}$$

From Equations (4-43)and (4-44) we have

$$A(s) = P(s,o) \tag{4-45}$$

$$B(s) = \frac{1}{x}\,[P(s,x) - P(s,o)] \tag{4-46}$$

$$C(s) = Q(s,o) \tag{4-47}$$

$$D(s) = \frac{1}{x}\,[Q(s,x) - Q(s,o)] \tag{4-48}$$

With the aid of these equations the logarithmic sensitivity function may be expressed directly in terms of the polynomials of the original transfer function.

$$S(s,x) = S_x^T = \frac{\partial \ln T(s,x)}{\partial \ln x} = \frac{x}{T(s,x)}\;\frac{\partial T(s,x)}{\partial x} \tag{4-49}$$

But

$$\frac{x}{T(s,x)} = \frac{x\,Q(s,x)}{P(s,x)} \tag{4-50}$$

and

$$\frac{\partial T(s,x)}{\partial x} = \frac{\partial}{\partial x}\left[\frac{A(s) + xB(s)}{C(s) + xD(s)}\right] \tag{4-51}$$

$$= \frac{B(s)C(s) - D(s)A(s)}{[Q(s,x)]^2}$$

Substituting Equations (4-50) and (4-51) into (4-49) one gets

$$S(s,x) = S_x^T = x\left[\frac{B(s)C(s) - A(s)D(s)}{P(s,x)\,Q(s,x)}\right]$$

$$= \frac{xB(s)Q(s,o)}{P(s,x)Q(s,x)} - \frac{xA(s)D(s)}{P(s,x)Q(s,x)}$$

$$= \frac{[P(s,x) - P(s,o)]\;Q(s,o) - [Q(s,x)]}{P(s,x)Q(s,x)}$$

$$= \left[\frac{Q(s,o)}{Q(s,x)} - \frac{P(s,o)}{P(s,x)}\right] \tag{4-52}$$

Now if the variable x changes by an incremental amount x, then an approximation given by the first terms of a Taylor series, T(s,x), changes by an incremental amount given by

$$\begin{aligned} \Delta T(s,x) &= T(s,x+\Delta x) - T(s,x) \\ &= \frac{\partial T(s,x)}{\partial x} \Delta x \\ &= \left(\frac{\Delta x}{x}\right) T(s,x)S(s,x) \qquad (4\text{-}53) \end{aligned}$$

For a given input function R(s), the system normal response is given by $$Y(s,x) = R(s)\, T(s,x)$$

The change in responce, Y(s,x), caused by the increment Δx is given by $$\Delta Y(s,x) = R(s)\,\Delta T(s,x) \qquad (4\text{-}54)$$

Substituting Equation (4-53) for $\Delta T(s,x)$, Equation (4-54) becomes

$$\Delta Y(s,x) = \left(\frac{\Delta x}{x}\right) R(s)T(s,x)S(s,x) \qquad (4\text{-}55)$$

Equation (4-55) indicates that $\Delta Y(s,x)$, the increment in the response, may be obtained by calculating the response due to an equivalent input perturbation signal, $\Delta R(s)$, applied to the original transfer function, T(s,x), where

$$\Delta R(s) = \left(\frac{\Delta x}{x}\right) R(s)\; S(s,x) \qquad (4,56)$$

<u>Simulation</u>: Equation (4-55) may be represented in block diagram form as drawn in Figure 4-3, where S(s,x) has been represented according to (4-52). From Equation (4-44), polynomials Q(s,x) and Q(s,o) can be written in expanded form as follows:

$$\begin{aligned} Q(s,x) &= (c_o + xd_o) + (c_1 + xd_1)s + \text{---------} \\ &\qquad + (c_{m-1} + xd_{m-1})s^{m-1} + s^m \qquad (4\text{-}57) \end{aligned}$$

$$Q(s,o) = c_o + c_1 s + \text{--------} + c_{m-1}s^{m-1} + s^m \qquad (4\text{-}58)$$

Simulation of the transfer function $Q(s,o)/Q(s,x)$ may be accomplished on an analogue computer as illustrated in Figure 4-4. A similar simulation is possible for the transfer function $P(s,o)/P(s,x)$ so that interconnection according to Figure 4-4 gives an analogue computer method for obtaining the change in the response, $\Delta Y(t,x)$, directly as a time function trace for any input signal.

Sinusoidal Steady State.

In the sinusoidal steady state, $s = j\omega$,

$$T(j\omega,x) = T(\omega,x) \quad e^{j\theta(\omega,x)} \tag{4-59}$$

where,

$$\tan^{-1}\theta = \frac{\operatorname{Im}T(j\omega,x)}{\operatorname{Re}T(j\omega,x)} \tag{4-60}$$

and

$$\ln T(j\omega,x) = \ln|T(\omega,x)| + j\theta(\omega,x) \tag{4-61}$$

From Equation (4-61) and the definition of $S(j\omega,x)$ as given by Equation (4-49) it follows that:

$$\operatorname{Re}[S(j\omega,x)] = \frac{\partial \ln T(\omega,x)}{\partial \ln x} \tag{4-62}$$

$$\operatorname{Im}[S(j\omega,x)] = \frac{\partial \theta(\omega,x)}{\partial \ln x} \tag{4-63}$$

Using Equations (4-62) and (4-63), if then x changes by an increment, Δx, the corresponding incremental changes, $\Delta \ln|T(\omega,x)|$, and $\Delta\theta(\omega,x)$, are given by

$$\frac{\Delta \ln|T(\omega,x)|}{\left(\frac{\Delta x}{x}\right)} = \operatorname{Re}[S(j\omega,x)] \tag{4-64}$$

$$\frac{\Delta\theta(\omega,x)}{(\Delta x/x)} = \operatorname{Im}[S(j\omega,x)] \tag{4-65}$$

IV-4 - Sensitivity Integrals

The expression for sensitivity leads to functions of parameter x (or k) and s, t or ω. From an engineering point of view it would be convenient to have an overall measure of sensitivity which is independent of either time or frequency. Such a figure would be useful for comparing the relative sensitivities of different parameters in a system as well as presenting the possibility of including a sensitivity criteria for purposes of design or optimization.

(a) - <u>Phase Integral</u>.

Consider first the area of the equivalent input perturbation signal, $\Delta r(t,x)$, relative to the area of the input signal, r(t). The total area of the input signal is given by the integral,

$$I_i = \lim_{t\to\infty} \int_0^t r(t)dt = \lim_{t\to\infty} \mathcal{L}^{-1}\left[\frac{R(s)}{s}\right]$$

$$= \lim_{s\to 0} \frac{sR(s)}{s} = R(0) \tag{4-66}$$

In the above calculation use is made of the fact that,

$$\int_0^t f(t)dt = \mathcal{L}^{-1}\left[\frac{F(s)}{s}\right]$$

and the final value theorem is applied. Similarly the total value of the area of the equivalent input perturbation signal is given by

$$\Delta I_i = \lim_{t\to\infty} \int_0^t \Delta r(t)dt$$

$$= \lim_{t\to\infty} \left(\frac{\Delta x}{x}\right) \mathcal{L}^{-1} \frac{R(s)S(s,x)}{S}$$

$$= \left(\frac{\Delta x}{x}\right) \lim_{s\to 0} \frac{s}{s}R(s)S(s,x)$$

$$= \left(\frac{\Delta x}{x}\right) R(o)S(o,x) \tag{4-68}$$

If $I_i \neq 0$, then the ratio of Equation (4-67) to Equation (4-66) denoted by I(x) is,

$$I(x) = \frac{\Delta I_i}{I_i} = \left(\frac{\Delta x}{x}\right) S(o,x) \qquad (4\text{-}68)$$

When $I_i = 0$ it means $R(0) = 0$ and therefore from Equation (4-67) it follows that, $\Delta I_i = 0$, provided $S(o,x) \neq \infty$. When $R(0) = 0$ and $S(o,x) = \infty$, then ΔI_i may be evaluated by calculating the limit of the product R(s) S(s,x) as indicated by Equation (4-67).

Alternately I(x) may be obtained by considering the area of the change in the response due to the increment Δx. Prior to the parameter change the area of the response is given by

$$\begin{aligned} I_o &= \lim_{t\to\infty} \int_o^t y(t,x)dt \\ &= \lim_{t\to\infty} \mathcal{L}^{-1} \frac{R(s)\ T(s,x)}{s} \\ &= \lim_{s\to o} \frac{s}{s}\ R(s)\ T(s,x) \\ &= R(o)\ T(o,x) \end{aligned} \qquad (4\text{-}69)$$

After the increment x, the area of the change in the response is with the aid of Equation (4-54) given by

$$\begin{aligned} \Delta I_o &= \lim_{t\to\infty} \int_o^t \Delta y(t,x)dt \\ &= \left(\frac{\Delta x}{x}\right) \lim_{t\to\infty} \mathcal{L}^{-1} \frac{R(s)T(s,x)S(s,x)}{s} \\ &= \left(\frac{\Delta x}{x}\right) \lim_{s\to o} \frac{s}{s}\ R(s)\ T(s,x)\ S(s,x) \\ &= \left(\frac{\Delta x}{x}\right) R(0)\ T(o,x)\ S(o,x) \end{aligned} \qquad (4\text{-}70)$$

If $I_o \neq 0$, the ratio of Equation (4-70) to Equation (4-69) exists and is given by

$$I(x) = \frac{\Delta I_o}{I_o} = \left(\frac{\Delta x}{x}\right) S(o,x) \tag{4-71}$$

When $I_o = 0$ it follows from Equation (4-80) that $I_o = 0$, provided $S(o,x) \neq 0$. When $I_o = 0$ and $S(o,x) = \infty$ then ΔI_o may be evaluated by calculating the limit of the product $R(s)\ T(s,x)\ S(s,x)$ as shown by Equation (4-70).

Now consider sinusoidal steady state and the area of the incremental phase characteristic $I_\theta(x,\omega)$ plotted on a logarithmic frequency scale, defined by the integral

$$I_\theta(\omega,x) = \int_{-\infty}^{\infty} \Delta\, \theta(\omega,x)\, d(\ln \omega)$$

$$= \int_{-\infty}^{\infty} \Delta\, \theta(\omega,x)\ \frac{d\omega}{\omega} \tag{4-72}$$

Substituting Equation (4-65) into Equation (4-72) we get

$$I_\theta(\omega,x) = \int_{-\infty}^{\infty} \left(\frac{\Delta x}{x}\right) \ \mathrm{Im}[S(j\omega,x)]\ \frac{d\omega}{\omega} = \left(\frac{\Delta x}{x}\right) \int_{-\infty}^{\infty} \mathrm{Im}[S(j\omega,x)]\ \frac{d\omega}{\omega} \tag{4-73}$$

From the reactance integral theorem (Appendix IV), it is known that

if $$H(j\omega) = R(\omega) + jx(\omega)$$

and $$\lim_{\omega \to \infty} H(j\omega) = 0$$

then $$\int_{-\infty}^{\infty} x(\omega)\ \frac{d\omega}{\omega} = -\frac{\pi}{2} \lim_{s \to 0} H(s) \tag{4-74}$$

Applying Equation (4-74) to Equation (4-73) we get

$$I_\theta(x,\omega) = -\frac{\pi}{2} \left(\frac{\Delta x}{x}\right) \lim_{s \to 0} S(s,x)$$

$$= -\frac{\pi}{2} \left(\frac{\Delta x}{x}\right) S(o,x)$$

$$= -\frac{\pi}{2}\ I(x) \tag{4-75}$$

Equations (4-68) , (4-71), and (4-75) enable the following theorem to be stated.

<u>Theorem</u> - For an incremental parameter change, Δx, the product of $\left(\frac{\Delta x}{x}\right)$ times the logarithmic sensitivity function evaluated at S = 0 is equal to

(a) - The ratio of the area of the equivalent input perturbation signal to the area of the input signal, provided the latter area is non-zero.

(b) - The fractional change in the system's response to any input signal, provided area of the response is non-zero.

(c) - $-\frac{2}{\pi}$ times the change in area of the sinusoidal steady state phase characteristic plotted on a logarithmic frequency scale.

When S(o,x) = 0, Equation (4-75) indicates that the increment Δx, causes zero net change in the area of the sinusoidal steady state phase characteristic plotted versus $\ln \omega$. Similarly, when S(o,x) = 0, Equation (4-70) indicates that Δx provides zero net change in the area of any time response characteristic which initially has finite none-zero area.

If the area of response is zero and S(o,x) is finite there will also be zero net change in area produced by Δx. S(o,x) may be calculated directly from Equation (4-52).

$$S(o,x) = \frac{Q(o,o)}{Q(o,x)} - \frac{P(o,o)}{P(o,x)} \tag{4-76a}$$

$$= \frac{c_o}{c_o + xd_o} - \frac{a_o}{a_o + xb_o} \tag{4-76b}$$

where, $Q(o,x) = c_o + xd_o$

$$P(o,x) = a_o + xb_o \tag{4-76b}$$

From Equation (4-76b) it is obvious that S(o,x) will be identically zero when $a_o d_o = c_o b_o$.

(b) - <u>The Magnitude Integral</u>.

It is also of interest to investigate the area of the change in the sinosoidal steady state magnitude characteristic as given by Equation (4-64). Consider the integral,

$$\Delta I_M(\omega,x) = \int_0^{\infty} \Delta \ln T(\omega,x) \; d\omega$$

$$= \left(\frac{\Delta x}{x}\right) \int_0^{\infty} \mathrm{Re}\,[S(j\omega,x)]\,d\omega \tag{4-77}$$

From the resistance integral theorem (Appendix IV) it is shown that if

$$H(j\omega) = R(j\omega) + jx(\omega) \tag{4-78}$$

and

$$\lim_{\omega \to \infty} H(j\omega) = 0$$

Then

$$\int_0^{\infty} R(\omega)\; d\omega = \frac{\pi}{2} \lim_{s \to \infty} sH(s) \tag{4-79}$$

Substituting Equation (4-79) into Equation (4-77) we get

$$\Delta I_M(\omega,x) = \left(\frac{\Delta x}{x}\right) \frac{\pi}{2} \lim_{s \to \infty} sS(s,x)$$

The use of this integral is illustrated in the following discussion.

(c) - <u>Sensitivity Squared Integral</u>.

In many cases S(o,x) is zero so it is not always a convenient criterion for comparing the sensitivities of different parameters in a system. This suggests the possibility of investigating the areas of the incremental error signals squared.

In previous sections of this chapter it was shown how the criterion of minimum square error to the incremental response was utilized. A simpler and less restrictive approach is to investigate the area of the equivalent input perturbation signal squared. Thus, the following integral is defined,

$$I^2(x) = \int_0^{\infty} [\Delta r(t,x)]^2 dt = \int_{-\infty}^{\infty} [\Delta r(t,x)]^2 dt \qquad (4\text{-}81)$$

where it is assumed that $\Delta r(t,x) = 0$ for $t < 0$. Applying Parceval's theorem to Equation (4-81), with the aid of Equation (4-56) we get

$$I_{\delta}^2(x) = \left(\frac{\Delta x}{x}\right)^2 \int_{-j\infty}^{j\infty} [R(s)S(s,x)]\,[R(-s)S(-s,x)]ds \qquad (4\text{-}82)$$

when $R(s) = 1$, $r(t) = \delta(t)$, then Equation (4-82) becomes

$$I_{\delta}^2(x) = \left(\frac{\Delta x}{x}\right)^2 \int_{-j\infty}^{j\infty} S(s,x)S(-s,x)ds \qquad (4\text{-}83)$$

Integrals of this type are tabluated in the literature (reference 2) and form the basis of analytical design procedures. In particular this integral can form a basis of comparison for the effects of different parameter variations upon system performance. The approach is illustrated by a simple example.

<u>Example:</u>

Consider a second order positional servomechanism subjected to a load disturbance torque $T_L(t)$ applied to the torque shaft. The transfer function relating the output angle to the applied torque is given by

$$\frac{\theta_o(s)}{T_L(s)} = \frac{1}{Js^2 + Fs + K} \qquad (4\text{-}84)$$

Where J, F and K are moment of inertia, viscous damping and gain

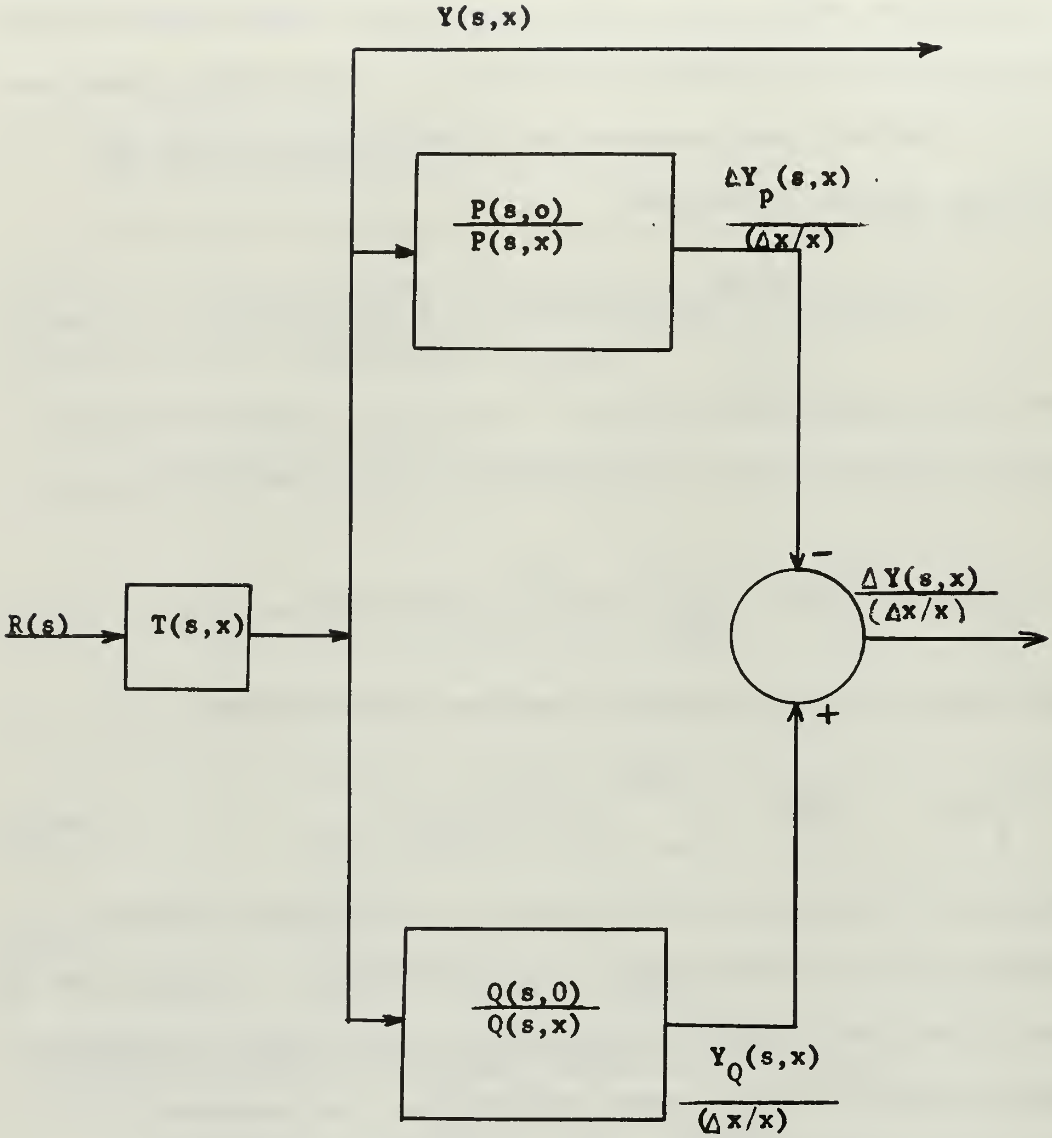


Fig. 4-3: Block Diagram for Generating the Incremental Response Caused by the Parameter Change, Δx.

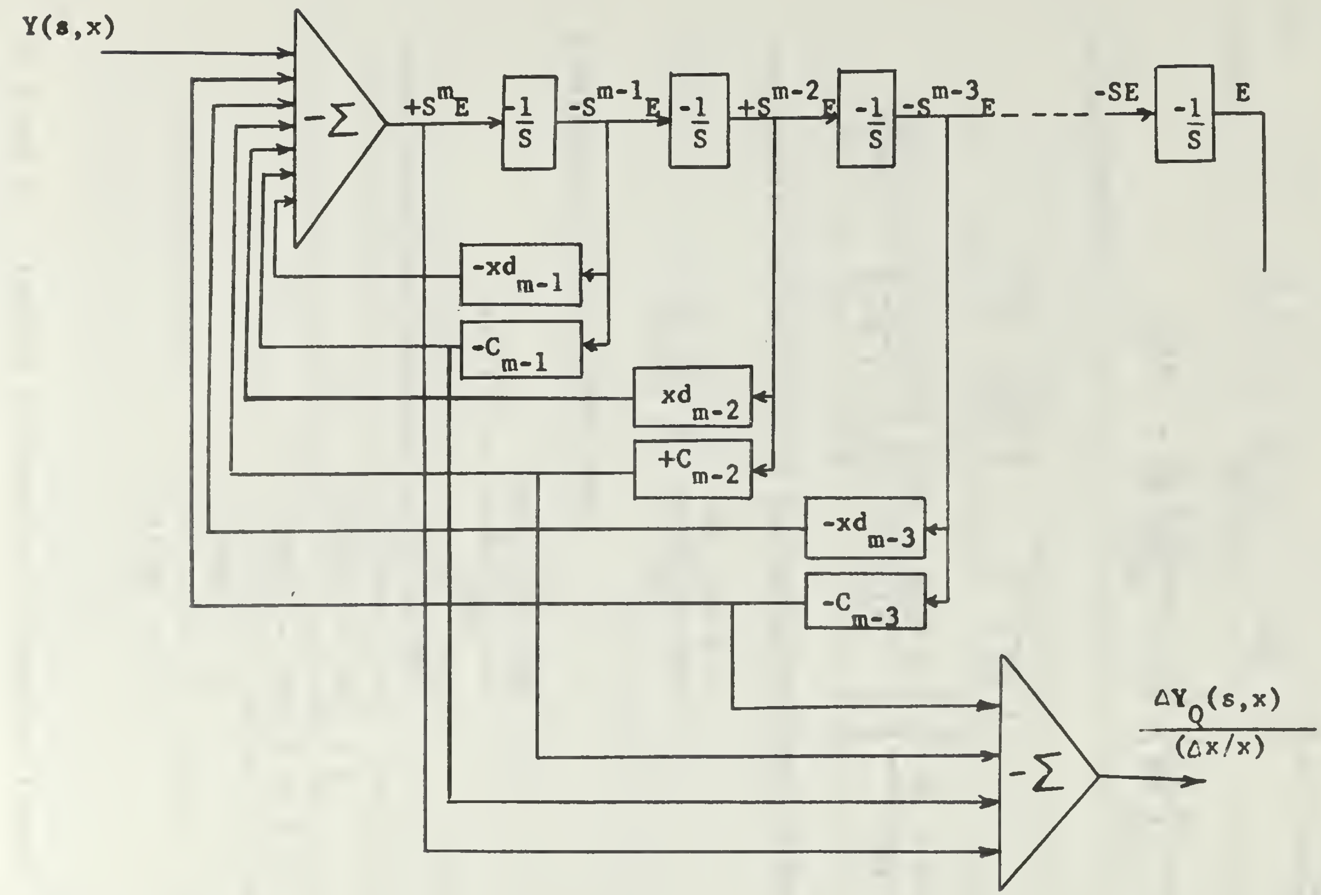


Fig. 4-4: Simulation of the Transfer Function

$$\frac{Q(s,0)}{Q(s,x)} = \left(\frac{E}{Q(s,x)}\right) \frac{Q(s,0)}{E}$$

respectively. It is desired to compare the sensitivity of the impulsive response when $T_L(t) = \delta(t)$ for incremental variations of F and K. From the polynomials of Equation (4-84), it follows:

$$x = F$$

$$P(s,F) = 1$$

$$P(s,o) = 1$$

$$Q(s,F) = Js^2 + Fs + K$$

$$Q(s,o) = Js^2 + K$$

Substituting into Equation (4-52), we get

$$S(s,F) = \frac{-Fs}{Js^2 + Fs + K} \tag{4-85}$$

Similarly it can be shown that

$$S(s,K) = \frac{-K}{Js^2 + Fs + K} \tag{4-86}$$

Substituting Equation (4-85) and Equation (4-83) we get

$$I_\delta^2(F) = \left(\frac{\Delta F}{F}\right)^2 \int_{-j\infty}^{j\infty} \left[\frac{-Fs}{Js^2 + Fs + K}\right] \left[\frac{Fs}{Js^2 - Fs + K}\right] ds$$

$$= \frac{F}{2J} \left(\frac{\Delta F}{F}\right)^2 = \zeta\omega_n \left(\frac{\Delta F}{F}\right)^2 \tag{4-87}$$

and similarly

$$I_\delta^2(k) = \frac{K}{2F} \left(\frac{\Delta K}{K}\right)^2 = \frac{\omega_n}{4\zeta} \left(\frac{\Delta K}{K}\right)^2 \tag{4-88}$$

where $\omega_n = \sqrt{K/J}$ and $\zeta = \dfrac{F}{2\sqrt{KJ}}$

ω_n is the undamped resonant frequency and ζ is the damping ratio. Equation (4-87) and (4-88) show that both sensitivity squared integrals are proportional to ω_n. For $\zeta > \frac{1}{2}$ a fractional variation of F is seen to have a greater effect upon its sensitivity squared

integral than the identical fractional variation of K has upon its integral.

IV-5 - Signal Flow Graph and Sensitivity

The general formulation of the transmittance of a signal flow graph described by Mason, can be extended to evaluate the sensitivity function. Mason has derived a simple and very useful rule for evaluating system transfer functions (or sometimes called transmittances) from signal flow graph. The transmission from any independent node x, called source, to a dependent node y, called sink, is

$$T_{xy} = \frac{\sum P_{xy} \Delta_{xy}}{\Delta}$$

Where the system determinent Δ is

$$\Delta = 1 - \sum_i L_i + \sum_{i,j} L_i' L_j' - \sum_{i,j,k} L_i'' L_j'' L_k'' + ---- \quad (4\text{-}90)$$

In Equation (4-90) L_i represents a loop transmission, i.e. the transmission around any closed-loop and $\sum_i L_i$ is the sum of all loop transmissions. $L_i' L_j'$ is the product of the loop transmissions of any two loops that do not have a node or branch in common, and $\sum_{i,j} L_i' L_j'$ is therefore the sum of all such products. $L_i'' L_j'' L''_k$ is the product of the loop transmissions of any three loops that do not have any branches or nodes in common.

The definiton of $P_{xy} \Delta_{xy}$ in Equation (4-89) is as follows: P_{xy} is any direct transmission from the input x to the output y. Each Δ_{xy} is associated with a particular P_{xy}. The value of Δ_{xy} is the same as Δ except for the removal of all terms containing any branches or nodes that appear in P_{xy}.

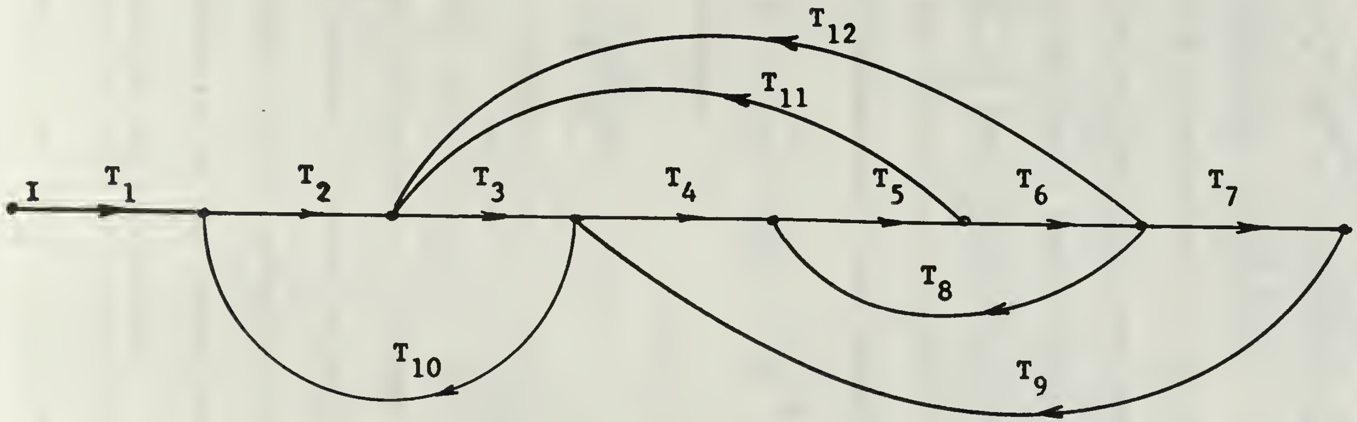


Fig. 4-5: A Signal Flow Graph for Applicaton of Mason's Gain Relation.

<u>**Example**</u>:

Find Δ, P_{io}, Δ_{io} and T_{io} in Figure 4-5. The sum of the individual loop transmissions is

$$L_i = T_2\, T_3\, T_{10} + T_3\, T_4\, T_5\, T_{11} + T_3\, T_4\, T_5\, T_6\, T_{12} + T_4\, T_5\, T_6\, T_7\, T_9 + T_5\, T_6\, T_8 \tag{4-91}$$

The above is best done in a systematic manner by starting with T_1 and listing all loops containing T_1, then going on to T_2, and taking all loops containing T_2 that have not already been listed, etc. Next we find $\sum_{i,j} L_i' L_j'$ which is the sum of the products of the loop transmissions taken two at a time, omitting all those that have any node or branch in common. To evaluate $\sum_{i,j} L_i' L_j'$, the list in $\sum_i L_i$ is examined. The first term in $\sum_i L_i$ is $T_2T_3T_{10}$. The other terms are scanned to detect those that do not contain any of T_2, T_3, or T_{10}. The first term satisfying this condition is $T_4T_5T_6T_7T_9$, but Figure 4-5 is examined to check whether these loops ($T_2T_3T_{10}$ and $T_4T_5T_6T_7T_9$) have any node in common. They do, and therefore their product is not included in $L_i' L_j'$. However the last term $T_5T_6T_8$ has no branches in common with $T_2T_3T_{10}$ and therefore $T_2T_3T_{10}T_5T_6T_8$ appears in $\sum L_i' L_j'$. Next we note second term in L_i, i.e., $T_3T_4T_5T_{11}$ and examine as above. In this way it is found that,

$$\sum_{i,j} L_i' L_j' = (T_2T_3T_{10})\,(T_5T_6T_8) \tag{4-92}$$

Since $\sum L_i' L_j'$ contains only one term, it is impossible for $\sum L_i'' L_j''$ to exist. Then

$$\Delta = 1 - \sum_i L_i + \sum_{i,j} L_i' L_j'$$

$$= 1 - [T_2T_3T_{10} + T_3T_4T_5T_{11} + T_3T_4T_5T_6T_{12} + T_4T_5T_6T_7T_9 + T_5T_6T_8] + T_2T_3T_{10}T_5T_6T_8 \tag{4-93}$$

Continuing with evaluation of P_{io} and Δ_{io} we have,

$$P_{io} = T_1T_2T_3T_4T_5T_6T_7 \tag{4-94}$$

Since each loop in ΣL_i has at least one branch present in P_{io}, then

$$\Delta_{io} = 1 \tag{4-95}$$

and

$$P_{io}\Delta_{io} = T_1T_2T_3T_4T_5T_6T_7 \tag{4-96}$$

Substituting Equation (4-93) and Equation (4-89) we get T_{io}.

<u>Formulation of Sensitivity Function</u>:

Sensitivity of a source to sink transmittance T_{ij} with respect to a parameter is defined as

$$S_x^{T_{ij}} = \frac{\partial \ln T_{ij}}{\partial \ln x} = \frac{\partial T_{ij}/T_{ij}}{\partial x/x} \tag{4-97}$$

As discussed in chapter 3, the sensitivity function may be expressed in the following form:

$$S_x^{T_{ij}} = \frac{1}{F_x}\left[1 + \frac{T_{ij}(x=0)}{T_{ij}}\right] = \frac{1}{F_x} - \frac{1}{F'_x} \tag{4-98}$$

Where $T_{ij}(x=0)$ is the transmittance from source i to sink j when the specified element x vanishes, i.e., when the branch x of the signal flow graph is removed and where F_x and F'_x are the return difference and the null return difference respectively, both referred to the element x. Furthermore,

$$T_{ij} = \frac{\Sigma P_{ij}\Delta_{ij}}{\Delta} \tag{4-99}$$

as discussed in previous sections. Expressing Equation (4-99) in logarithmic form and differentiating with respect to $\ln x$, we get

$$S_x^{T_{ij}} = \frac{\partial \ln \Sigma(P_{ij}\Delta_{ij})}{\partial \ln x} - \frac{\partial \ln \Delta}{\partial \ln x} \tag{4-100}$$

The terms on the right hand side are themselves sensitivity functions

and one may therefore regard the sensitivity function of Equation (4-97) as a difference of simpler sensitivity functions. $\frac{\partial \ln \Delta}{\partial \ln x}$ is the sensitivity of the graph determinent with respect to the element, x. The way the determinent Δ was defined, it is then only a function of the feedback branch, taking the form of the sum of the products of branch transmittances, none of which appear more than once in any single product. Therefore we can write Δ as a linear function of s,

$$\Delta = ax + \Delta^o$$

and

$$\frac{\partial \Delta}{\partial x} = a$$

or

$$x \frac{\partial \Delta}{\partial x} = ax = \Delta - \Delta^o \qquad (4\text{-}101)$$

Where Δ^o is the determinant when $x = 0$, i.e., when the specified branch x of the signal flow graph is removed. However accordinging to signal flow graph theory,

$$F_x = \frac{\Delta}{\Delta^o} \qquad (4\text{-}102)$$

Dividing Equation (4-101) by Δ we get

$$S_x = \frac{\partial \Delta/\Delta}{\partial x/x} = 1 - \Delta^o/\Delta = 1 - \frac{1}{F_x} \qquad (4\text{-}103)$$

Similarly, the first term of the right hand side of Equation (4-100), which may be thought of as the path sensitivity, is found to be

$$S_x^P = 1 - \frac{\sum (P_{ij} \Delta_{ij})^o}{\sum (P_{ij} \Delta_{ij})} = 1 - \frac{1}{F_x} \qquad (4\text{-}104)$$

Then

$$S_x^{T_{ij}} = S_x^P - S_x^\Delta = \frac{1}{F_x} - \frac{1}{F'_x} \qquad (4\text{-}105)$$

Equation (4-105) was previously obtained in chapter 2. Also sensitivity could be written as,

$$S_x^{T_{ij}} = \frac{\Delta^o}{\Delta} - \frac{\sum (P_{ij} \quad \Delta_{ij})^o}{\sum (P_{ij} \quad \Delta_{ij})} \tag{4-106}$$

In certain configurations this may be an easy way to evaluate sensitivity functions. However this is recommended as an interesting topic for further investigation to utilize the concept of signal flow graph and its related techniques.

IV-6 - Multiparameter Sensitivity

So far we discussed the sensitivity of a transmission function, T, which depends upon a single parameter x and defined as

$$S_x^T = \frac{d \ln T}{d \ln x} \tag{4-107}$$

Now consider T as a function of n parameters, i.e.,

$$T = T\,(x_1, x_2, \ldots\ldots, x_n)$$

Several definitons have been proposed. Hakimi and Cruz (see reference 18) define multiparameter sensitivity by ,

$$0 \leq \overset{\text{maximum}}{\Delta x_i} \leq \delta_i \quad \left(\frac{\Delta T}{\Delta T} \Big/ \prod_i \; \delta_i \right) \tag{4-108}$$

where δ_i is the maximum variation of the ith parameter as specified in particular application. For small δ_i, the fractional change in transmission is given by (neglecting second order effects),

$$\frac{\Delta T}{T} = d(\ln T) = \sum_i \frac{\partial(\ln T)}{\partial(\ln x_i)} \; d(\ln x_i) \tag{4-109}$$

Setting $y_i = \ln x_i$, we can consider the set of fractional parameter increments as a vector,

$$dy = [d(\ln x_1), \text{-------------}, d(\ln x_n)]$$

and Equation (4-109) becomes the scalar product

$$\frac{\Delta T}{T} = \nabla \left[\ln T(y_1, \text{----------} y_n) \right] \cdot dy \tag{4-110}$$

The sensitivity of transmission clearly depends upon the gradient vector $\nabla \ln T$. Kuo and Goldstein (reference 18) suggest that multiparameter sensitivity S^T be defined as

$$S^T = \nabla \left[\ln T(y_1, \text{-------}, y_n) \right] \tag{4-111}$$

This has the advantage that it agrees with the single parameter definition in Equation (4-107). The magnitude of the multiparameter sensitivity function is

$$\left| S^T \right| = (S^T \cdot S^{-T})^{1/2} \tag{4-112}$$

Donald A. Calohan (reference 17) defines the parameter sensitivity vector,

$$S_x^{T(j\omega)} = \sum_i \frac{\partial \ln T(j\omega)}{\partial \ln x_i} \hat{x}_i \tag{4-113}$$

$$= \sum_i \frac{\partial \alpha(\omega)}{\partial x_i} x_i \hat{x}_i + j \sum_i \frac{\partial \beta(\omega)}{x_i} x_i \hat{x}_i$$

where

$\alpha(\omega) = \ln \left| T(j\omega) \right|$, the attenuation

$\beta(\omega) = \arg T(j\omega)$, the phase

x_i = element varying

$\hat{x}_i$ = a set of orthogonal unit vectors

Hence,

$$\sum_i \frac{\partial \alpha}{\partial x_i} x_i \hat{x}_i = \mathrm{Re}\,(S_x^{T(j\omega)}) \tag{4-114}$$

$$\sum_i \frac{\partial \beta}{x_i} x_i \hat{x}_i = \mathrm{Im}\,(S_x^{T(j\omega)} \tag{4-115}$$

The magnitude of the vector $\mathrm{Re}(S_x^{T(j\omega)})$, i.e.,

$$\left| \text{Re}\ (S_x^{T(j\omega)}) \right| = \sqrt{\sum_i \left(\frac{\partial \alpha(\omega)}{\partial x_i}\ x_i \right)^2} \qquad (4\text{-}116)$$

is therefore an indication of the sensitivity. Multiparameter sensitivity could best be discussed in terms of state space, and is recommended as a topic for further work on this line.

V

CONCLUSION

The object of this thesis as stated in the "Introduction," was a literary research on the subject of sensitivity applied to the linear feedback control system, with only one parameter changing. Two different cases were discussed. One when the parameter changes are incremental and the other when parameter changes are moderate or large. In each case a system sensitivity was defined associated with methods of measuring this sensitivity function.

In the case of incremental parameter variation, the concept of root sensitivity was presented and from that a graphical design technique discussed. When the parameter variations were large, a new definition of sensitivity was introduced and it was shown that when the leakage transmission between the input and output is negligible the two definitions are identical. A design philosophy, and the use of Nichols Chart was also presented. Finally, the new aspects of sensitivity such as specification for sensitivity function based on forced response criterion and least mean square error were explained. A new approach based on the signal perturbation theory together with sensitivity integrals were also introduced.

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Appendix I

Bilinear Theorem, Return Difference, Null Return Difference

1 - The Fundamental Feedback Flow Graph.

The signal flow graph for a linear feedback control system is shown in Figure A1-1, where,

t_{oi} is transmission from input I to output O

t_{os} is transmission from source S to output O

t_{ci} is transmission from input I to control C

t_{cs} is transmission from source S to control C.

Then we can write,

$$\text{Output } O = t_{oi}I + t_{os}S$$

and

$$S = kt_{ci}/(1 - kt_{cs})$$

Hence,

$$\frac{O}{I} \overset{D}{=} T(s) = t_{oi} + \frac{kt_{ci}t_{os}}{1 - kt_{cs}} \qquad \text{(A1-1)}$$

Equation (A1-1) is called the "fundamental feedback equation" and is the most important equation in linear feedback theory and all the significant properties of linear feedback systems are derivable from this equation. As an analytic tool it shows how a linear circuit problem can be broken up into two simple circuits.

2 - Return Difference.

The system transfer function as derived above is

$$T(s) = \frac{O}{I} = t_{oi} + kt_{ci}t_{os}/(1 - kt_{cs})$$

suppose it is asked "what is the effect of the feedback around k, on the system function T?" Clearly the effect is given by the quantity $1 - kt_{cs}$. This quantity $1 - kt_{cs}$ has a very simple physical interpretation readily seen by opening the closed loop anywhere, for example

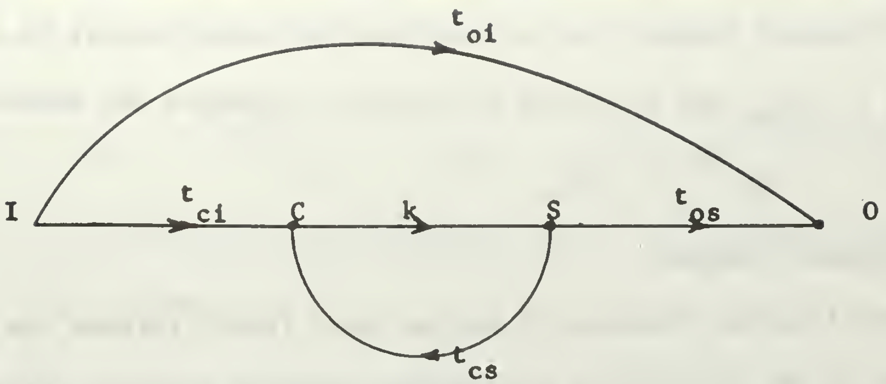

Fig. A1-1: Fundamental Feedback Flow Graph.

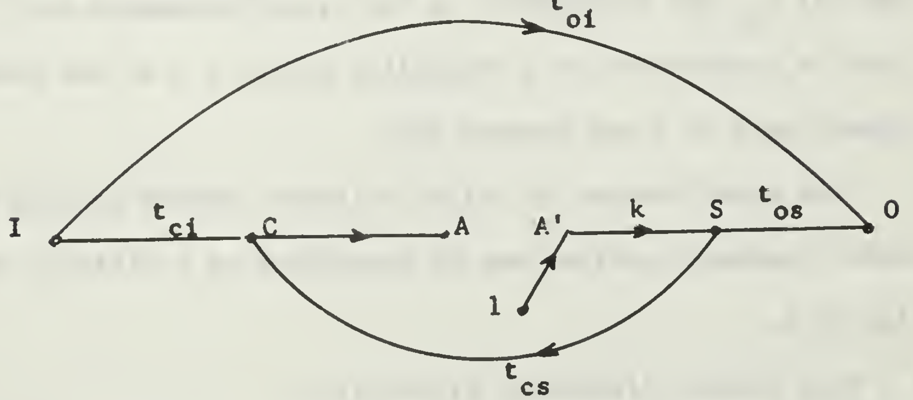

Fig. A1-2: Physical interpretation of loop transmission function.

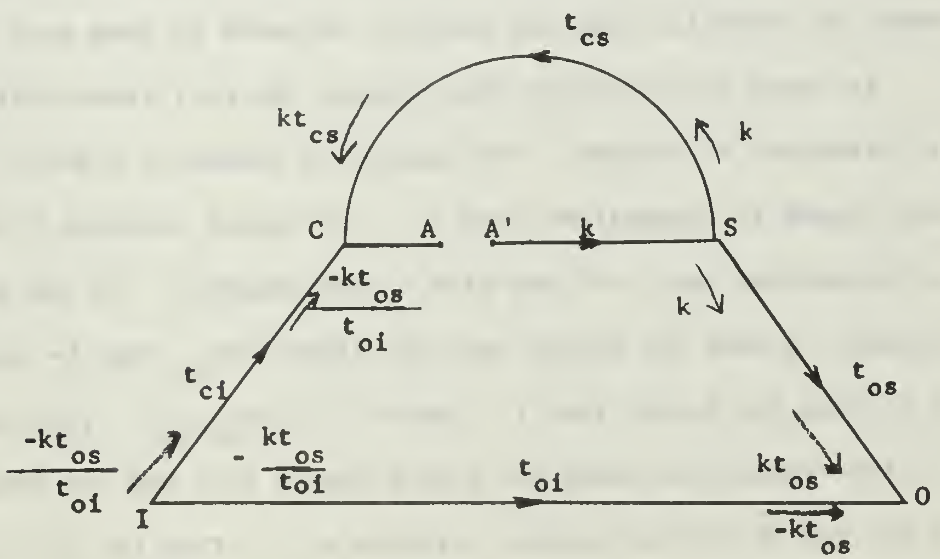

Fig. A1-3: Derivation of null return difference.

at AA' and measure the signal returned at A, the latter is kt_{cs}. The difference between the injected and the return signal is precisely $1 - kt_{cs}$ and is called the return difference for element k.

$$F_k \overset{D}{=} 1 - kt_{cs} \qquad \text{(A1-2)}$$

3 - Bilinear Theorem:

The transfer function of any two port linear systems can be written in the form of the fundamental feedback equation (A1-1), such that all t_{ij} are independent of the system parameter k(s), providing k can be represented as a controlled source S = kC and providing k appears only in S and nowhere else.

The above theorem is called bilinear theorem because the fundamental feedback equation may be considered as a bilinear transformation of k.

4 - Null Return Difference, Figure A1-2.

The null return difference is the return difference evaluated under the condition that the input is adjusted to give zero output.

In terms of the signal flow diagram the null return difference is determined as follows. The diagram is broken at A and A' and a unit signal is transmitted from A'. The signal reaching S is k. This is transmitted back to C and also to the output O. If the input is adjusted to make the output zero the signal $-kt_{os}$ must be arriving at O along the branch from I. Hence I is $-kt_{os}/t_{oi}$. This input is also transmitted along the branch from I to C and the return signal is the sum of the two signals arriving at C. Then the null return difference with reference to k is given by the relation,

$$F_k' = 1 - (kt_{cs} - kt_{os}t_{ci}/t_{oi})$$

The system transfer function can be written in terms of return difference and null return difference,

$$T = t_{oi} + \frac{kt_{ci}t_{os}}{1 - kt_{cs}} = t_{oi}\left[\frac{1 - kt_{cs} + kt_{ci}t_{os}/t_{oi}}{1 - kt_{cs}}\right] \tag{A1-3}$$

Where F_k' is the null return difference and F_k is the return difference with reference to k.

Appendix II

The Curvature of the Root Locus at Ordinary Points

The curvature of a curve, which is defined by means of an implicit relationship $F(x,y) = 0$ is given by

$$\frac{1}{\rho} = - \frac{F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2}{(F_x^2 + F_y^2)^{3/2}} \tag{A2-1}$$

where

$$F_x = \frac{\partial F}{\partial x} \tag{A2-2}$$

$$F_y = \frac{\partial F}{\partial y} \tag{A2-3}$$

$$F_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial x^2} \tag{A2-4}$$

$$F_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial^2 F}{\partial x \partial y} \tag{A2-5}$$

$$F_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial^2 F}{\partial y^2} \tag{A2-6}$$

If we introduce a new cartisian system with its origin at $\bar{s}_j$ and oriented so that the tangent to the locus coincides with the u-axis. Then in this system for the implicit function $F(u,v)$ we have (Figure A2-1)

$$\left.\frac{\partial F}{\partial u}\right|_{\substack{v=0\\u=0}} = F_u(0,0) = 0$$

and

$$\frac{1}{\rho} = - \frac{F_{uu}F_v^2 - 2F_{uv}F_uF_v + F_{vv}F_u^2}{(F_u^2 + F_v^2)^{3/2}} \tag{A2-7}$$

and since $F_u(0,0) = 0$ and curvature is evaluated at point (0,0), then

$$\frac{1}{\rho} = - \left.\frac{F_{uu}}{F_v}\right|_{\substack{u=0\\v=0}}$$

The center of curvature is on the line normal to the curve. Because of the perpendicular choice of coordinate the positive sign of ρ means

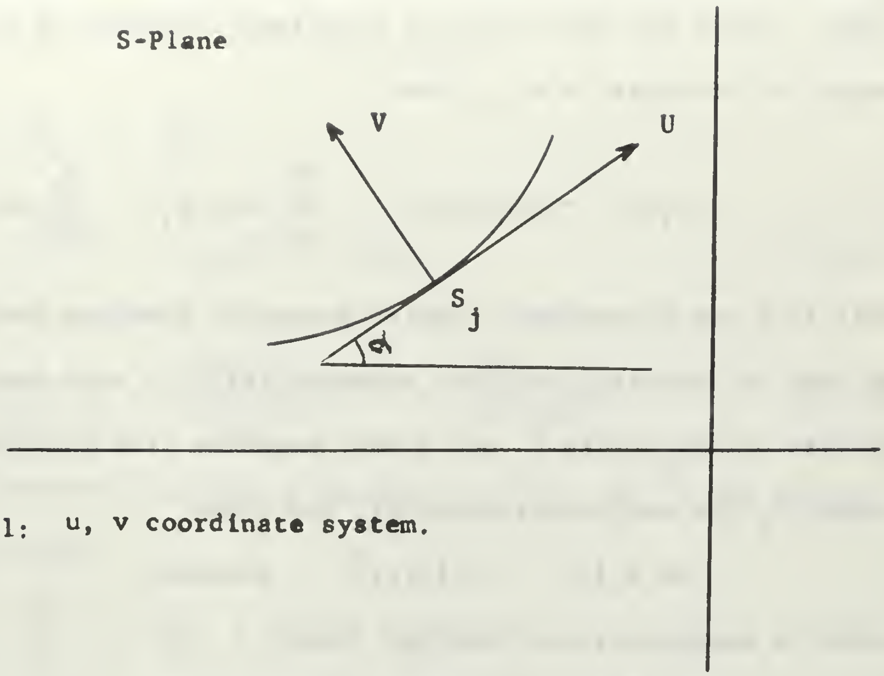

Fig. A2-1: u, v coordinate system.

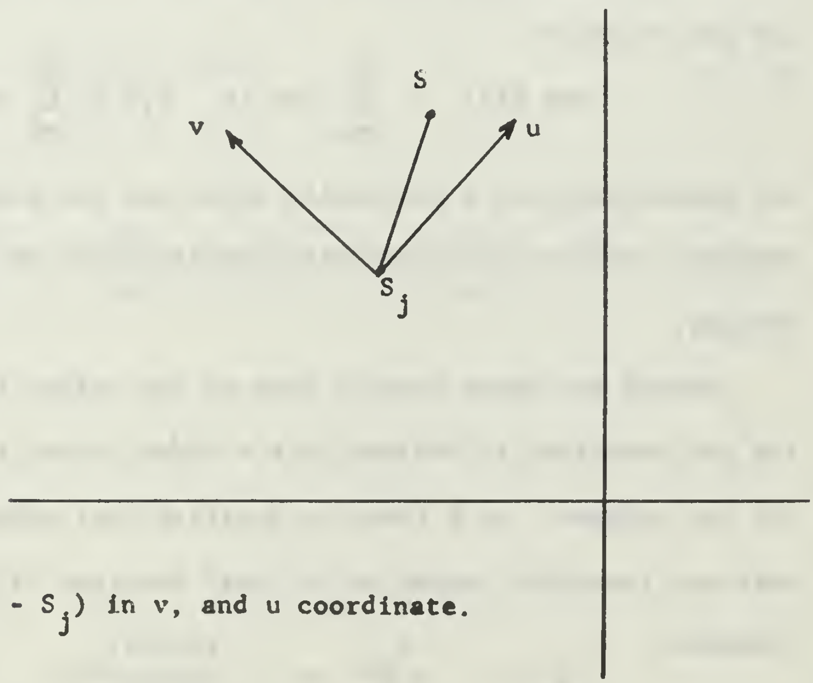

Fig. A2-2: $(S - S_j)$ in v, and u coordinate.

that the center of curvature lies on the positive v-axis and vice versa. Since the gain, K(s), is a rational function of s and analytic except at its poles $s = z_i$, then

$$K_1(s) = \ln[K(s)] = \sum_{j=1}^{n} \ln(s-P_j) - \sum_{i=1}^{m} \ln(s-Z_i) + \pi j \tag{A2-9}$$

This is a two dimensional complex potential function such as would be used to describe the field around a set of n unit positive line charges at the points P_j and m unit negative line charges at the points Z_i for such configurations, the lines

$$\operatorname{Re} K_1(s) = \ln|K(s)| = \text{constant} \tag{A2-10}$$

would be equipotentials, and the lines

$$\operatorname{Im} K_1(s) = \arg K(s) = \text{constant} \tag{A2-11}$$

would be stream-lines or lines of force. Since the above relations and the relation

$$\arg K(s) = \sum_{i=1}^{m} \arg(s - Z_i) - \sum_{j=1}^{n} \arg(s - P_i) \tag{A2-12}$$

are equivalent with a particular value for the constant, it follows that the root loci coincide with certain lines of force in potential analogy.

Having the above idea in mind we can define function F as $F = \operatorname{Im}K$ and sometimes is defined as $F = \operatorname{Im}\ln K$, which is more suitable for our purpose. As K takes on positive real values, F is equal to zero and therefore serves as the real function of Equation (A2-1).

Therefore

$$\frac{1}{\rho} = -\frac{F_{uu}}{F_v} = -\frac{(\operatorname{Im}\ln K)_{uu}}{(\operatorname{Im}\ln K)_v}$$

$$= -\frac{\operatorname{Im}\left[(\ln K)_{uu}\right]}{\operatorname{Im}\left[(\ln K)_v\right]} \tag{A2-13}$$

In general K is expressed as a function of the variable $Z = u + jv$ and since lnK is an analytic function then,

$$\frac{\partial}{\partial u} = \frac{\partial}{\partial Z}$$

$$\frac{\partial}{\partial v} = j\frac{\partial}{\partial jv} = j\frac{d}{dZ} \qquad \text{(A2-14)}$$

Hence

$$\frac{1}{\rho} = -\frac{\mathrm{Im}(\ln K)_{zz}}{\mathrm{Im}j(\ln K)_z} = -\frac{\mathrm{Im}(\ln K)_{zz}}{\mathrm{Re}(\ln K)_z} \qquad \text{(A2-15)}$$

To evaluate Equation (A2-15) we assume $Z = (s - s_o)e^{-j\gamma}$, and for any function f we have

$$\frac{df}{dZ} = \frac{df}{ds} \cdot \frac{ds}{dZ} = \frac{df}{ds}\, e^{j\gamma}$$

$$\frac{d^2f}{dZ^2} = \frac{d^2f}{ds^2}\left(\frac{ds}{dZ}\right)^2 + \frac{df}{ds}\,\frac{d^2s}{dZ^2}$$

$$= \frac{d^2f}{ds^2}\, e^{2j\gamma} \qquad \text{(A2-16)}$$

and similarly

$$\frac{d^nf}{dZ^n} = \frac{d^nf}{ds^n}\left(\frac{ds}{dZ}\right)^n = \frac{d^nf}{ds^n}\, e^{jn\gamma} \qquad \text{(A2-17)}$$

Appendix III

Derivation of an Expression for the Sensitivity of a Multiple Order System Root

If we write the expansion of the total differential of L(s) to include higher-order terms (see reference 15, page 81), we have

$$(dL)_{s = s_j} = 0 = \left[\frac{\partial L}{\partial s} ds + \frac{\partial L}{\partial K} dK + \sum_i \frac{\partial L}{\partial Z_i} dZ_i + \sum_i \frac{\partial L}{\partial P_i} dP_i\right]$$

$$+ \frac{1}{2!}\left[\text{same as above bracket}\right]^2$$

$$+ \frac{1}{3!}\left[\text{-----------}\right] + \text{-----------}$$

where

$$\left[\frac{\partial L}{\partial s} ds + \frac{\partial L}{\partial K} dK\right]^a = \frac{\partial^a L}{\partial s^a}(ds)^a$$

$$+ a\frac{\partial^a L}{\partial s^{a-1}\partial K}(ds)^{a-1} dK + \frac{a(a-1)}{2!}\frac{\partial^a L}{\partial s^{a-2}\partial K^2}(ds)^{a-2}(dK)^2$$

$$+ \text{------------} + \frac{\partial^a K}{\partial K^a}(dK)^a$$

Next retain only the lowest order terms for each parameter and note that the first (N-1) derivatives of L with respect to S are zero at $S = S_j$. Then

$$dS_j = \left[(-1)^{N+1} N!\right]^{\frac{1}{N}}\left[\frac{\partial L}{\partial K} dK + \sum_i \frac{\partial L}{\partial Z_i} dZ_i + \sum_i \frac{\partial L}{\partial P_i} dP_i\right]^{1/N}_{s=s_j}$$

This suggests the following notation:

$$dS_j = \left[S_K^{s_j}\frac{dK}{K} + \sum_i S_{Z_i}^{s_j} dZ_i + \sum_i S_{P_i}^{s_j} dP_i\right]^{1/N}$$

and comparing the above two equation we have

$$S_K^{s_j} = \frac{(-1)^N \, N!}{\left[\frac{\partial^N L}{\partial s} \right]_{s = s_j}}$$

Appendix IV

Resistance and Reactance Integrals

To prove the resistance and reactance integrals, we start with Cauchy's residue theorem. Consider any region defined by a closed boundary in which and on whose boundary H(s) is single valued and analytic, except for poles of any finite numbers of multiplicity. Then the line integral of H(s) around this boundary is related to the residues of those poles of H(s) which are located inside the region as follows,

$$\oint H(s)ds = 2\pi j \sum \text{residues}.$$

If H(s) has no singularities in the right half plane and as on Figure A4-1, then

$$\oint H(s)\,ds = \lim_{R\to\infty} \int_{-jR}^{jR} H(s)d(j\omega) + \lim_{R\to\infty} \int_{\pi/2}^{-\pi/2} H(R,\theta)d(Re^{j\theta})$$

$$= I_1 + I_2 = 0 \qquad \text{(A4-1)}$$

Next we consider some properties of the loop transmission function H(s). The loop transmission function can be regarded as transforms of impulse responses of real systems. The transform of a time function is

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt \qquad \text{(A4-2)}$$

So

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\,[\cos\omega t - j\sin\omega t]\,dt$$

$$= R(\omega) + jx(\omega) \qquad \text{(A4-3)}$$

where

$$R(\omega) = \int_{-\infty}^{\infty} h(t)\cos\omega t\,dt \text{ is an even function of } \omega$$

and

$$x(\omega) = -\int_{-\infty}^{\infty} h(t)\sin\omega t\,dt \text{ is an odd function of } \omega .$$

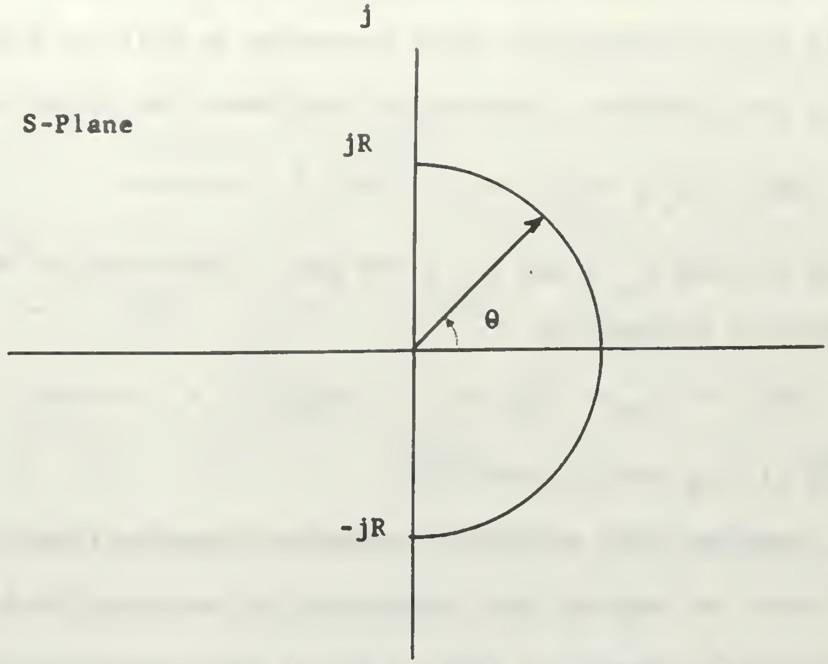

Fig. A4-1: Application of Cauchy's residue theorem.

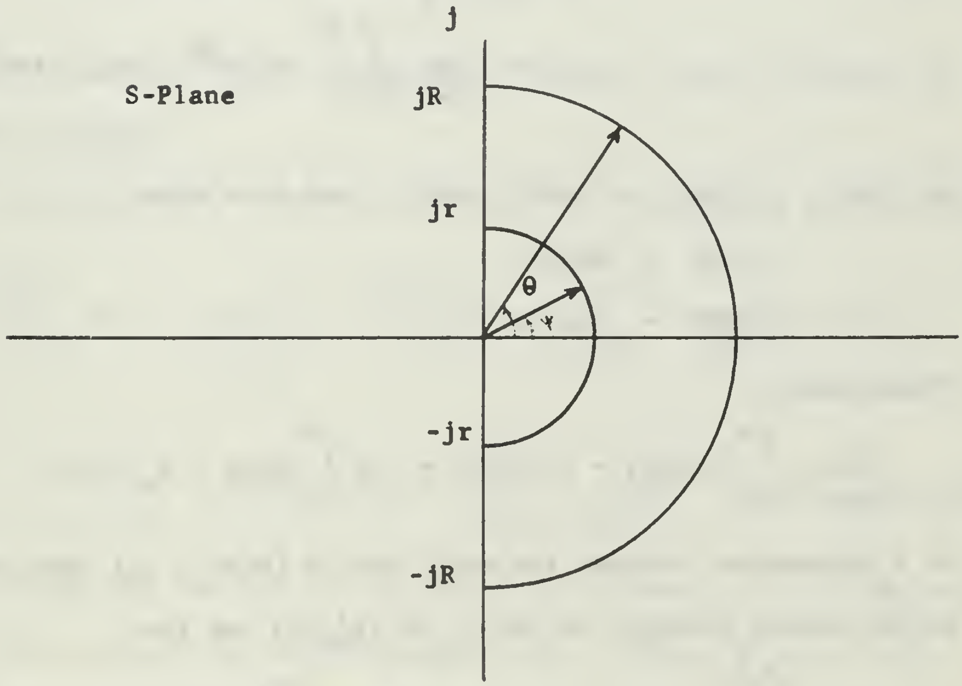

Fig. A4-2: Contour for F(s)/s.

Thus, $H(j\omega) = R(\omega) + Jx(\omega) = \overline{H(-j\omega)} = A(-\omega) - Jx(-\omega)$ (A4-4)

It is often necessary to focus attention on H(s) for s near zero and for s near infinity. Expressing H(s) about the origin gives

$$H(s) = A_o + B'_o s + A''_o s^2 + B''_o s^3 + \text{--------} \tag{A4-5}$$

where all the A'_o's and B'_o's are real. The notation for the expansion of H(s) at infinity is

$$H(s) = A_\infty + (B'_\infty/s) + (A''_\infty/s^2) + \text{--------} \tag{A4-6}$$

where all A'_∞ and B'_∞ are real.

Consider H(s) so that it satisfies Equation (A4-4), is finite (or zero) at infinity and is analytic in the right half plane. Applying Cauchy's theorem to $[H(s) - H(\infty)]$ over the right half plane whose boundary is the $j\omega$-axis and the right half-infinite semi-circle,

$$\begin{aligned} \oint [H(s) - H(\omega)]ds &= 0 \\ &= \lim_{R\to\infty} \int_{-R}^{R} [H(j\omega) - A_\infty]\, jd\omega \\ &\quad + \lim_{R\to\infty} \int_{-\pi/2}^{\pi/2} [H(Re^{j\theta}) - A_\infty]\, d(Re^{j\theta}) \end{aligned} \tag{A4-7}$$

But let $H(j\omega) = R(\omega) + jx(\omega)$, and note that,

$$R(\omega) = R(-\omega)$$

$$x(\omega) = -x(-\omega)$$

Consequently,

$$\lim_{R\to\infty} \int_{-R}^{R} [H(j\omega) - A_\infty]\, jd\omega = j2\int_0^\infty [R(\omega) - A_\infty]\, d\omega$$

As s approaches infinity the only term of $[H(s) - A_\infty]$ that contributes to the second integral in (A4-7) is (B'_∞/s) and then

$$\lim_{R\to\infty} \int_{-\pi/2}^{\pi/2} [H(s) - A_\infty]ds = \lim_{R\to\infty} \int_{\pi/2}^{-\pi/2} B'_\infty/s \; ds$$

$$= \lim_{R\to\infty} \int_{\pi/2}^{-\pi/2} B'_{\infty}/s \quad Re^{j\theta}\, d\theta$$

$$= \lim_{R\to\infty} \int_{\pi/2}^{-\pi/2} B'_{\infty}\, jd\theta \quad = \quad -j\pi B'_{\infty}$$

and therefore,

$$j2 \int_0^{\infty} [R(\omega) - A_{\infty}]\, d\omega - j\pi B'_{\infty} = 0$$

or

$$\int_0^{\infty} [R(\omega) - A_{\infty}]\, d\omega = 1/2\pi B'_{\infty} \qquad \text{(A4-8)}$$

Note, that if we had used H(s) rather than $[H(s) - A_{\infty}]$, as the integrand, then the integral will be infinite unless $A_{\infty} = 0$. If

$$\lim_{s\to\infty} H(s) = A_{\infty} = 0$$

then

$$B'_{\infty} = \lim_{s\to\infty} SH(s)$$

and

$$\int_0^{\infty} R(\omega)\, d\omega = 1/2\pi \lim_{s\to\infty} SH(s) \qquad \text{(A4-9)}$$

This is the simple form of "Resistance Integral" used in Chapter IV.

<u>Reactance Integral</u>:

We apply Cauchy's theorem to H(s)/S over the contour Figure A4-2

$$\oint \frac{H(s)}{s}\, ds = \lim_{\substack{r\to 0\\ R\to\infty}} \left\{ \int_r^R \frac{H(j\omega)d\omega}{\omega} + \int_{\pi/2}^{-\pi/2} \frac{A_{\infty}}{Re^{j\theta}}\, d(Re^{j\theta}) \right.$$

$$\left. + \int_{-R}^{r} \frac{H(j\omega)}{\omega}\, d\omega + \int_{-\pi/2}^{\pi/2} \frac{A_o}{re^{j\psi}}\, d(re^{j\psi}) \right\} \qquad \text{(A4-10)}$$

When the first and the third term of the above equation are combined, the odd parts cancel and the even parts add. Writing $H(j\omega) = R(\omega) + jx(\omega)$, Equation (A4-10) reduces to

$$2 \int_{-\infty}^{\infty} x(\omega) \frac{d\omega}{\omega} + \pi (A_0 - A_\infty) = 0$$

or

$$\int_{-\infty}^{\infty} x(\omega) \frac{d\omega}{\omega} = -\pi/2 \, (A_0 - A_\infty) \qquad \text{(A4-11)}$$

If A = 0, then Equation (A4-11) reduces to

$$\int_{-\infty}^{\infty} x(\omega) \frac{d\omega}{\omega} = -\pi/2 \lim_{s \to 0} H(s)$$

which is the simple form of reactance integral used in Chapter IV.

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This thesis is the result of literary research on the subject of sensitivity in linear feedback control systems. It is a synopsis of information obtained from various technical publications and is designed to give the reader information on the theory of sensitivity and its application in design problems. Included are

(1) Definiton of sensitivity function and root sensitivity for incremental variations of signle parameter and their design application.
(2) Definition of sensitivity function applied to the large parameter variations and its design application.
(3) Special topics such as: Specification for sensitivity function, sensitivity integrals, and introduction to multiparameter sensitivity.

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